LOCAL CONVERGENCE OF SPECTRA AND PSEUDOSPECTRA

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Abstract. We prove local convergence results for the spectra and pseudospectra of sequences of linear operators acting in different Hilbert spaces and converging in generalised strong resolvent sense to an operator with possibly non-empty essential spectrum. We establish local spectral exactness outside the limiting essential spectrum, local ε-pseudospectral exactness outside the limiting essential ε-near spectrum, and discuss properties of these two notions including perturbation results.

1. Introduction

We address the problem of convergence of spectra and pseudospectra for a sequence $(T_n)_{n\in\mathbb{N}}$ of closed linear operators approximating an operator T. We establish regions $K \subset \mathbb{C}$ of local convergence,

$$\lim_{n\to\infty} \sigma(T_n) \cap K = \sigma(T) \cap K, \tag{1.1}$$

$$\lim_{n\to\infty} \overline{\sigma_\varepsilon(T_n)} \cap K = \overline{\sigma_\varepsilon(T)} \cap K, \quad \varepsilon > 0, \tag{1.2}$$

where the limits are defined appropriately. Recall that for $\varepsilon > 0$ the ε-pseudospectrum is defined as the open set

$$\sigma_\varepsilon(T) := \left\{ \lambda \in \mathbb{C} : \|(T-\lambda)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

employing the convention that $\|(T-\lambda)^{-1}\| = \infty$ for $\lambda \in \sigma(T)$ (see [35] for an overview).

We allow the operators T, T_n, $n \in \mathbb{N}$, to act in different Hilbert spaces H, H_n, $n \in \mathbb{N}$, and require only convergence in so-called generalised strong resolvent sense, i.e. the sequence of projected resolvents $((T_n-\lambda)^{-1}P_{H_n})_{n\in\mathbb{N}}$ shall converge strongly to $(T-\lambda)^{-1}P_H$ in a common larger Hilbert space.

The novelty of this paper lies in its general framework which is applicable to a wide range of operators T and approximating sequences $(T_n)_{n\in\mathbb{N}}$: 1) We do not assume selfadjointness as in [29, Section VIII.7], [37, Section 9.3], or boundedness of the operators as in [34, 36, 17, 13] (see also [14] for an overview). 2) The operators may have non-empty essential spectrum, in contrast to the global spectral exactness results for operators with compact resolvents [3, 27, 38]. 3) The results are applicable, but not restricted to, the domain truncation method for differential operators [26, 9, 10, 11, 28, 15] and to the Galerkin (finite section) method [24, 8, 32, 25, 5, 7, 6]. 4) Our assumptions are weaker than the convergence in operator norm [21] or in (generalised) norm resolvent sense [4, 20].

Regarding convergence of spectra (see (1.1)), the aim is to establish *local spectral exactness* of the approximation $(T_n)_{n\in\mathbb{N}}$ of T, i.e.

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(1) *local spectral inclusion*: For every $\lambda \in \sigma(T) \cap K$ there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of $\lambda_n \in \sigma(T_n) \cap K$, $n \in \mathbb{N}$, with $\lambda_n \to \lambda$ as $n \to \infty$;
(2) *no spectral pollution*: If there exists a sequence $(\lambda_n)_{n\in I}$ of $\lambda_n \in \sigma(T_n) \cap K$, $n \in I$, with $\lambda_n \to \lambda$ as $n \in I$, $n \to \infty$, then $\lambda \in \sigma(T) \cap K$.

Concerning pseudospectra (see (1.2)), we define *local ε-pseudospectral exactness, -inclusion, -pollution* in an analogous way by replacing all spectra by the closures of ε-pseudospectra.

In general, spectral exactness is a major challenge for non-selfadjoint problems. In the selfadjoint case, it is well known that generalised strong resolvent convergence implies spectral inclusion, and if the resolvents converge even in norm, then spectral exactness prevails [37, Section 9.3]. In the non-selfadjoint case, norm resolvent convergence excludes spectral pollution; however, the approximation need not be spectrally inclusive [23, Section IV.3]. Stability problems are simpler when passing from spectra to pseudospectra; in particular, they converge (ε-pseudospectral exactness) under generalised norm resolvent convergence [4, Theorem 2.1]. However, if the resolvents converge only strongly, ε-pseudospectral pollution may occur.

In the two main results (Theorems 2.3, 3.6) we prove local spectral exactness outside the *limiting essential spectrum* $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big)$, and local ε-pseudospectral exactness outside the *limiting essential ε-near spectrum* $\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$. The notion of limiting essential spectrum was introduced by Boulton, Boussaïd and Lewin in [8] for Galerkin approximations of selfadjoint operators. Here we generalise it to our more general framework,

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) := \left\{\lambda \in \mathbb{C} : \begin{array}{l} \exists\, I \subset \mathbb{N}\, \exists\, x_n \in \mathcal{D}(T_n),\ n \in I, \text{ with} \\ \|x_n\| = 1,\ x_n \xrightarrow{w} 0,\ \|(T_n - \lambda)x_n\| \to 0 \end{array}\right\},$$

and further to pseudospectral theory,

$$\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) := \left\{\lambda \in \mathbb{C} : \begin{array}{l} \exists\, I \subset \mathbb{N}\, \exists\, x_n \in \mathcal{D}(T_n),\ n \in I, \text{ with} \\ \|x_n\| = 1,\ x_n \xrightarrow{w} 0,\ \|(T_n - \lambda)x_n\| \to \varepsilon \end{array}\right\}.$$

Outside these problematic parts, we prove convergence of the (ε-pseudo-) spectra with respect to the Hausdorff metric. In the case of pseudospectra, the problematic part is the whole complex plane if T has constant resolvent norm on an open set (see Theorem 3.8 and also [4]).

The paper is organised as follows. In Section 2 we study convergence of spectra. First we prove local spectral exactness outside the limiting essential spectrum (Theorem 2.3). Then we establish properties of $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}$, including a spectral mapping theorem (Theorem 2.5) which implies a perturbation result for $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big)$ (Theorem 2.12). In Section 3 we address pseudospectra and prove local ε-pseudospectral convergence (Theorem 3.6). Then we establish properties of the limiting essential ε-near spectrum $\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$ including a perturbation result (Theorem 3.15). In the final Section 4, applications to the Galerkin method of block-diagonally dominant matrices and to the domain truncation method of perturbed constant-coefficient PDEs are studied.

Throughout this paper we denote by H_0 a separable infinite-dimensional Hilbert space. The notations $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ refer to the norm and scalar product of H_0. Strong and weak convergence of elements in H_0 is denoted by $x_n \to x$ and $x_n \xrightarrow{w} 0$, respectively. The space $L(H)$ denotes the space of all bounded operators acting in a Hilbert space H. Norm and strong operator convergence in $L(H)$ is denoted by $B_n \to B$ and $B_n \xrightarrow{s} B$, respectively. An identity operator is denoted by I; scalar multiples λI are written as λ. Let $H, H_n \subset H_0$, $n \in \mathbb{N}$, be closed subspaces and $P = P_H : H_0 \to H$, $P_n = P_{H_n} : H_0 \to H_n$, $n \in \mathbb{N}$, be the orthogonal projections onto the respective subspaces and suppose that they converge strongly, $P_n \xrightarrow{s} P$.

Throughout, let T and T_n, $n \in \mathbb{N}$, be closed, densely defined linear operators acting in the spaces H, H_n, $n \in \mathbb{N}$, respectively. The domain, spectrum, point spectrum, approximate point spectrum and resolvent set of T are denoted by $\mathcal{D}(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{\mathrm{app}}(T)$ and $\varrho(T)$, respectively, and the Hilbert space adjoint operator of T is T^*. For non-selfadjoint operators there exist (at least) five different definitions for the essential spectrum which all coincide in the selfadjoint case; for a discussion see [16, Chapter IX]. Here we use

$$\sigma_{\mathrm{ess}}(T) := \Big\{\lambda \in \mathbb{C} : \exists\, (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T) \text{ with } \|x_n\| = 1,\, x_n \overset{w}{\to} 0,\, \|(T-\lambda)x_n\| \to 0\Big\},$$

which corresponds to $k = 2$ in [16]. The remaining spectrum $\sigma_{\mathrm{dis}}(T) := \sigma(T)\backslash\sigma_{\mathrm{ess}}(T)$ is called the discrete spectrum. For a subset $\Omega \subset \mathbb{C}$ we denote $\Omega^* := \{\overline{z} : z \in \Omega\}$. Finally, for two compact subsets $\Omega, \Sigma \subset \mathbb{C}$, their Hausdorff distance is $\mathrm{d_H}(\Omega, \Sigma) := \max\big\{\sup_{z\in\Omega} \mathrm{dist}(z, \Sigma), \sup_{z\in\Sigma} \mathrm{dist}(z, \Omega)\big\}$ where $\mathrm{dist}(z, \Sigma) := \inf_{w\in\Sigma} |z - w|$.

2. Local convergence of spectra

In this section we address the problem of local spectral exactness. In Subsection 2.1 we introduce the limiting essential spectrum $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}$ and state the main result (Theorem 2.3). Then we establish properties of $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}$ in Subsection 2.2, including a spectral mapping theorem (Theorem 2.5) which implies a perturbation result for $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big)$ (Theorem 2.12). At the end of the section, in Subsection 2.3, we prove the main result and illustrate it for the example of Galerkin approximations of perturbed Toeplitz operators.

2.1. Main convergence result.

The following definition of generalised strong and norm resolvent convergence is due to Weidmann [37, Section 9.3] who considers selfadjoint operators.

Definition 2.1. i) The sequence $(T_n)_{n\in\mathbb{N}}$ is said to *converge in generalised strong resolvent sense* to T, denoted by $T_n \overset{gsr}{\to} T$, if there exist $n_0 \in \mathbb{N}$ and $\lambda_0 \in \bigcap_{n\geq n_0} \varrho(T_n) \cap \varrho(T)$ with

$$(T_n - \lambda_0)^{-1} P_n \overset{s}{\longrightarrow} (T - \lambda_0)^{-1} P, \quad n \to \infty.$$

ii) The sequence $(T_n)_{n\in\mathbb{N}}$ is said to *converge in generalised norm resolvent sense* to T, denoted by $T_n \overset{gnr}{\to} T$, if there exist $n_0 \in \mathbb{N}$ and $\lambda_0 \in \bigcap_{n\geq n_0} \varrho(T_n) \cap \varrho(T)$ with

$$(T_n - \lambda_0)^{-1} P_n \longrightarrow (T - \lambda_0)^{-1} P, \quad n \to \infty.$$

The following definition generalises a notion introduced in [8] for the Galerkin method of selfadjoint operators.

Definition 2.2. The *limiting essential spectrum* of $(T_n)_{n\in\mathbb{N}}$ is defined as

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) := \left\{\lambda \in \mathbb{C} : \begin{array}{l} \exists\, I \subset \mathbb{N}\, \exists\, x_n \in \mathcal{D}(T_n),\, n \in I, \text{ with} \\ \|x_n\| = 1,\, x_n \overset{w}{\to} 0,\, \|(T_n - \lambda)x_n\| \to 0 \end{array}\right\}.$$

The following theorem is the main result of this section. We characterise regions where approximating sequences $(T_n)_{n\in\mathbb{N}}$ are locally spectrally exact and establish spectral convergence with respect to the Hausdorff metric.

Theorem 2.3. i) *Assume that $T_n \overset{gsr}{\to} T$ and $T_n^* \overset{gsr}{\to} T^*$. Then spectral pollution is confined to the set*

$$\sigma_{\mathrm{ess}}\left((T_n)_{n\in\mathbb{N}}\right) \cup \sigma_{\mathrm{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*, \tag{2.1}$$

and for every isolated $\lambda \in \sigma(T)$ that does not belong to the set in (2.1), there exists a sequence of $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, with $\lambda_n \to \lambda$, $n \to \infty$.

ii) *Assume that* $T_n \stackrel{gsr}{\to} T$ *and* T_n, $n \in \mathbb{N}$, *all have compact resolvents. Then claim* i) *holds with* (2.1) *replaced by the possibly smaller set*

$$\sigma_{\mathrm{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*. \tag{2.2}$$

iii) *Suppose that the assumptions of* i) *or* ii) *hold, and let* $K \subset \mathbb{C}$ *be a compact subset such that* $K \cap \sigma(T)$ *is discrete and belongs to the interior of* K. *If the intersection of* K *with the set in* (2.1) *or* (2.2), *respectively, is contained in* $\sigma(T)$, *then*

$$\mathrm{d_H}\big(\sigma(T_n)\cap K, \sigma(T)\cap K\big) \longrightarrow 0, \quad n\to\infty.$$

2.2. **Properties of the limiting essential spectrum.** In this subsection we establish properties that the limiting essential spectrum shares with the essential spectrum (see [16, Sections IX.1,2]).

The following result follows from a standard diagonal sequence argument; we omit the proof.

Proposition 2.4. *The limiting essential spectrum* $\sigma_{\mathrm{ess}}\left((T_n)_{n\in\mathbb{N}}\right)$ *is a closed subset of* $\mathbb{C}$.

The limiting essential spectrum satisfies a mapping theorem.

Theorem 2.5. *Let* $\lambda_0 \in \bigcap_{n\in\mathbb{N}} \varrho(T_n)$ *and* $\lambda \neq \lambda_0$. *Then the following are equivalent:*

(1) $\lambda \in \sigma_{\mathrm{ess}}\left((T_n)_{n\in\mathbb{N}}\right)$;
(2) $(\lambda-\lambda_0)^{-1} \in \sigma_{\mathrm{ess}}\big(\big((T_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}\big)$.

Proof. "(1) $\Longrightarrow$ (2)": Let $\lambda \in \sigma_{\mathrm{ess}}\left((T_n)_{n\in\mathbb{N}}\right)$. By Definition 2.2 of the limiting essential spectrum, there exist $I \subset \mathbb{N}$ and $x_n \in \mathcal{D}(T_n)$, $n \in I$, such that $\|x_n\| = 1$, $x_n \stackrel{w}{\to} 0$ and $\|(T_n-\lambda)x_n\| \to 0$. Note that $\|(T_n-\lambda_0)x_n\| \to |\lambda-\lambda_0| \neq 0$, hence there exists $N \in \mathbb{N}$ such that $\|(T_n-\lambda_0)x_n\| > 0$ for every $n \in I$ with $n \geq N$. Define

$$y_n := \frac{(T_n-\lambda_0)x_n}{\|(T_n-\lambda_0)x_n\|}, \quad n \in I, \quad n \geq N.$$

Then $\|y_n\| = 1$ and

$$y_n = \frac{(T_n-\lambda)x_n}{\|(T_n-\lambda_0)x_n\|} + \frac{(\lambda-\lambda_0)x_n}{\|(T_n-\lambda_0)x_n\|} \stackrel{w}{\longrightarrow} 0, \quad n\to\infty.$$

Moreover, we calculate

$$\begin{aligned}\left\|\left((T_n-\lambda_0)^{-1} - (\lambda-\lambda_0)^{-1}\right)y_n\right\| &= \left\|\frac{x_n - (\lambda-\lambda_0)^{-1}(T_n-\lambda_0)x_n}{\|(T_n-\lambda_0)x_n\|}\right\| \\ &= |\lambda-\lambda_0|^{-1}\frac{\|(T_n-\lambda)x_n\|}{\|(T_n-\lambda_0)x_n\|} \longrightarrow 0, \quad n\to\infty.\end{aligned}$$

This implies $(\lambda-\lambda_0)^{-1} \in \sigma_{\mathrm{ess}}\big(\big((T_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}\big)$.

"(2) $\Longrightarrow$ (1)": It is easy to check that if there exist an infinite subset $I \subset \mathbb{N}$ and $y_n \in H_n$, $n \in I$, with $\|y_n\| = 1$, $y_n \stackrel{w}{\to} 0$ and $\left\|\left((T_n-\lambda_0)^{-1} - (\lambda-\lambda_0)^{-1}\right)y_n\right\| \to 0$, then

$$x_n := \frac{(T_n-\lambda_0)^{-1}y_n}{\|(T_n-\lambda_0)^{-1}y_n\|}, \quad n \in I,$$

satisfy $\|x_n\| = 1$, $x_n \stackrel{w}{\to} 0$ and $\|(T_n-\lambda)x_n\| \to 0$. □

Remark 2.6. For the Galerkin method, Theorem 2.5 is different from the spectral mapping theorem [8, Theorem 7] for semi-bounded selfadjoint operators. Whereas in Theorem 2.5 the resolvent of the approximation, i.e. $(P_nT|_{\mathcal{R}(P_n)} - \lambda_0)^{-1}$, is considered, the result in [8] is formulated in terms of the approximation of the resolvent, i.e. $P_n(T-\lambda_0)^{-1}|_{\mathcal{R}(P_n)}$, which is in general not easy to compute.

The essential spectrum is contained in its limiting counterpart.

Proposition 2.7. i) *Assume that* $T_n \overset{gsr}{\to} T$. *Then* $\sigma_{\rm ess}(T) \subset \sigma_{\rm ess}\big((T_n)_{n\in\mathbb{N}}\big)$.
ii) *If* $T_n \overset{gnr}{\to} T$, *then* $\sigma_{\rm ess}(T) = \sigma_{\rm ess}\left((T_n)_{n\in\mathbb{N}}\right)$.

For the proof we use the following elementary result.

Lemma 2.8. *Assume that* $T_n \overset{gsr}{\to} T$. *Then for all* $x \in \mathcal{D}(T)$ *there exists a sequence of elements* $x_n \in \mathcal{D}(T_n)$, $n \in \mathbb{N}$, *with* $\|x_n\| = 1$, $n \in \mathbb{N}$, *and*

$$\|x_n - x\| + \|T_n x_n - Tx\| \longrightarrow 0, \quad n \to \infty. \tag{2.3}$$

Proof. By Definition 2.1 i) of $T_n \overset{gsr}{\to} T$, there exist $n_0 \in \mathbb{N}$ and $\lambda_0 \in \varrho(T)$ such that $\lambda_0 \in \varrho(T_n)$, $n \geq n_0$, and

$$(T_n - \lambda_0)^{-1} P_n \overset{s}{\longrightarrow} (T - \lambda_0)^{-1} P, \quad n \to \infty. \tag{2.4}$$

Let $x \in \mathcal{D}(T)$ and define

$$y_n := (T_n - \lambda_0)^{-1} P_n (T - \lambda_0) x \in \mathcal{D}(T_n), \quad n \geq n_0.$$

Then, using $P_n \overset{s}{\to} P$ and (2.4), it is easy to verify that $\|y_n - x\| \to 0$ and $\|T_n y_n - Tx\| \to 0$. In particular, there exists $n_1 \geq n_0$ such that $y_n \neq 0$ for all $n \geq n_1$. Now (2.3) follows for arbitrary normalised $x_n \in \mathcal{D}(T_n)$, $n < n_1$, and $x_n := y_n/\|y_n\|$, $n \geq n_1$. □

Proof of Proposition 2.7. i) Let $\lambda \in \sigma_{\rm ess}(T)$. By definition, there exist an infinite subset $I \subset \mathbb{N}$ and $x_k \in \mathcal{D}(T)$, $k \in I$, with $\|x_k\| = 1$, $x_k \overset{w}{\to} 0$ and

$$\|(T - \lambda) x_k\| \longrightarrow 0, \quad k \to \infty. \tag{2.5}$$

Let $k \in I$ be fixed. Since $T_n \overset{gsr}{\to} T$, Lemma 2.8 implies that there exists a sequence of elements $x_{k;n} \in \mathcal{D}(T_n)$, $n \in \mathbb{N}$, such that $\|x_{k;n}\| = 1$, $\|x_{k;n} - x_k\| \to 0$ and $\|T_n x_{k;n} - Tx_k\| \to 0$ as $n \to \infty$. Let $(n_k)_{k\in I}$ be a sequence such that $n_{k+1} > n_k$, $k \in I$, and

$$\|x_{k;n_k} - x_k\| < \frac{1}{k}, \quad \|T_{n_k} x_{k;n_k} - Tx_k\| < \frac{1}{k}, \quad k \in I. \tag{2.6}$$

Define $\widetilde{x}_k := x_{k;n_k} \in \mathcal{D}(T_{n_k})$, $\quad k \in I$. Then (2.6) and $x_k \overset{w}{\to} 0$ imply $\widetilde{x}_k \overset{w}{\to} 0$ as $k \in I$, $k \to \infty$. Moreover, (2.5) and (2.6) yield $\|(T_{n_k} - \lambda)\widetilde{x}_k\| \to 0$ as $k \in I$, $k \to \infty$. Altogether we have $\lambda \in \sigma_{\rm ess}\left((T_n)_{n\in\mathbb{N}}\right)$.

ii) The inclusion $\sigma_{\rm ess}(T) \subset \sigma_{\rm ess}\left((T_n)_{n\in\mathbb{N}}\right)$ follows from i). Let $\lambda \in \sigma_{\rm ess}\left((T_n)_{n\in\mathbb{N}}\right)$. By the assumption $T_n \overset{gnr}{\to} T$, there exist $n_0 \in \mathbb{N}$ and $\lambda_0 \in \varrho(T)$ with $\lambda_0 \in \varrho(T_n)$, $n \geq n_0$, and $(T_n - \lambda_0)^{-1} P_n \to (T - \lambda_0)^{-1} P$. The mapping result established in Theorem 2.5 implies $(\lambda - \lambda_0)^{-1} \in \sigma_{\rm ess}\big(\left((T_n - \lambda_0)^{-1}\right)_{n\geq n_0}\big)$. So there are an infinite subset $I \subset \mathbb{N}$ and $x_n \in H_n$, $n \in I$, with $\|x_n\| = 1$, $x_n \overset{w}{\to} 0$ and

$$\left\|\left((T_n - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1}\right) x_n\right\| \longrightarrow 0, \quad n \in I, \quad n \to \infty.$$

Moreover, in the limit $n \to \infty$ we have

$$\begin{aligned}\left\|\left((T - \lambda_0)^{-1} P - (\lambda - \lambda_0)^{-1}\right) x_n\right\| &\leq \left\|\left((T_n - \lambda_0)^{-1} P_n - (\lambda - \lambda_0)^{-1}\right) x_n\right\| \\ &\quad + \left\|(T_n - \lambda_0)^{-1} P_n - (T - \lambda_0)^{-1} P\right\| \longrightarrow 0.\end{aligned}$$

Hence

$$0 \neq (\lambda - \lambda_0)^{-1} \in \sigma_{\rm ess}\big((T - \lambda_0)^{-1} P\big) \subset \sigma_{\rm ess}\big((T - \lambda_0)^{-1}\big) \cup \{0\}.$$

Now $\lambda \in \sigma_{\rm ess}(T)$ follows from the mapping theorem [16, Theorem IX.2.3, k=2] for the essential spectrum. □

Now we study sequences of operators and perturbations that are compact or relatively compact in a sense that is appropriate for sequences. We use Stummel's notion of discrete compactness of a sequence of bounded operators (see [33, Definition 3.1.(k)]).

Definition 2.9. Let $B_n \in L(H_n)$, $n \in \mathbb{N}$. The sequence $(B_n)_{n\in\mathbb{N}}$ is said to be *discretely compact* if for each infinite subset $I \subset \mathbb{N}$ and each bounded sequence of elements $x_n \in H_n$, $n \in I$, there exist $x \in H$ and an infinite subset $\widetilde{I} \subset I$ so that $\|x_n - x\| \to 0$ as $n \in \widetilde{I}$, $n \to \infty$.

Proposition 2.10. i) *If $T_n \in L(H_n)$, $n \in \mathbb{N}$, are so that $(T_n)_{n\in\mathbb{N}}$ is a discretely compact sequence and $(T_n^* P_n)_{n\in\mathbb{N}}$ is strongly convergent, then*

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) = \{0\}.$$

If, in addition, $(T_n P_n)_{n\in\mathbb{N}}$ is strongly convergent, then

$$\sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* = \{0\}.$$

ii) *If there exists $\lambda_0 \in \bigcap_{n\in\mathbb{N}} \varrho(T_n)$ such that $((T_n - \lambda_0)^{-1})_{n\in\mathbb{N}}$ is a discretely compact sequence and $\big((T_n^* - \overline{\lambda_0})^{-1} P_n\big)_{n\in\mathbb{N}}$ is strongly convergent, then*

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) = \emptyset.$$

If, in addition, $\big((T_n - \lambda_0)^{-1} P_n\big)_{n\in\mathbb{N}}$ is strongly convergent, then

$$\sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* = \emptyset.$$

For the proof we need the following lemma. Claim ii) is the "discrete" analogue for operator sequences of the property of operators to be completely continuous.

Lemma 2.11. *Let $B_n \in L(H_n)$, $n \in \mathbb{N}$, and $B_0 \in L(H_0)$ with $B_n^* P_n \xrightarrow{s} B_0^*$. Consider an infinite subset $I \subset \mathbb{N}$ and elements $x \in H_0$ and $x_n \in H_n$, $n \in I$, such that $x_n \xrightarrow{w} x$ as $n \in I$, $n \to \infty$.*

i) *We have $x \in H$ and $B_n x_n \xrightarrow{w} B_0 x \in H$ as $n \in I$, $n \to \infty$.*

ii) *If $(B_n)_{n\in\mathbb{N}}$ is discretely compact, then $B_n x_n \to B_0 x$ as $n \in I$, $n \to \infty$.*

Proof. i) First note that, for any $z \in H_0$, we have

$$\langle x_n, z\rangle = \langle x_n, P_n z\rangle \longrightarrow \langle x, Pz\rangle = \langle Px, z\rangle, \quad n \in I, \quad n \to \infty,$$

and hence $x_n \xrightarrow{w} Px$. By the uniqueness of the weak limit, we obtain $x = Px \in H$. The weak convergence $B_n x_n \xrightarrow{w} B_0 x$ is shown analogously, and also $B_n x_n = P_n B_n x_n \xrightarrow{w} P B_0 x$ which proves $B_0 x = P B_0 x \in H$.

ii) Assume that there exist an infinite subset $I_0 \subset I$ and $\varepsilon > 0$ such that

$$\|B_n x_n - B_0 x\| > \varepsilon, \quad n \in I_0. \tag{2.7}$$

Since the sequence $(x_n)_{n\in I_0}$ is bounded and $(B_n)_{n\in\mathbb{N}}$ is a discretely compact sequence, by Definition 2.9 there exists an infinite subset $\widetilde{I} \subset I_0$ such that $(B_n x_n)_{n\in\widetilde{I}} \subset H_0$ is convergent (in H_0) to some $y \in H$. Then, by claim i), the strong convergence $B_n^* P_n \xrightarrow{s} B_0^*$ and the weak convergence $x_n \xrightarrow{w} x$ imply $B_n x_n \xrightarrow{w} B_0 x \in H$ as $n \in \widetilde{I}$, $n \to \infty$. By the uniqueness of the weak limit, we obtain $y = B_0 x$, and therefore $(B_n x_n)_{n\in\widetilde{I}}$ converges to $B_0 x$. The obtained contradiction to (2.7) proves the claim. □

Proof of Proposition 2.10. i) Let $I \subset \mathbb{N}$ be an infinite subset, and let $x_n \in \mathcal{D}(T_n)$, $n \in I$, satisfy $\|x_n\| = 1$ and $x_n \xrightarrow{w} 0$. Lemma 2.11 ii) implies $T_n x_n \to 0$. Now the first claim follows immediately.

Now assume that, in addition, $(T_nP_n)_{n\in\mathbb{N}}$ is strongly convergent. Then, by [3, Proposition 2.10], the sequence $(T_n^*)_{n\in\mathbb{N}}$ is discretely compact. Now the second claim follows analogously as the first claim.

ii) The assertion follows from i) and the mapping result in Theorem 2.5; note that $(\lambda-\lambda_0)^{-1}\neq 0$ for all $\lambda\in\mathbb{C}$. □

The limiting essential spectrum is invariant under (relatively) discretely compact perturbations.

Theorem 2.12. i) *Let $B_n\in L(H_n)$, $n\in\mathbb{N}$. If the sequence $(B_n)_{n\in\mathbb{N}}$ is discretely compact and $(B_n^*P_n)_{n\in\mathbb{N}}$ is strongly convergent, then*

$$\sigma_{\mathrm{ess}}\left((T_n+B_n)_{n\in\mathbb{N}}\right)=\sigma_{\mathrm{ess}}\left((T_n)_{n\in\mathbb{N}}\right).$$

If, in addition, $(B_nP_n)_{n\in\mathbb{N}}$ is strongly convergent, then

$$\sigma_{\mathrm{ess}}\left(((T_n+B_n)^*)_{n\in\mathbb{N}}\right)^*=\sigma_{\mathrm{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*.$$

ii) *For $n\in\mathbb{N}$ let A_n be a closed, densely defined operator in H_n. If there exists $\lambda_0\in\bigcap_{n\in\mathbb{N}}\varrho(T_n)\cap\bigcap_{n\in\mathbb{N}}\varrho(A_n)$ such that the sequence*

$$\big((T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}$$

is discretely compact and $\Big(\big((T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1}\big)^*P_n\Big)_{n\in\mathbb{N}}$ is strongly convergent, then

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big)=\sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big).$$

If, in addition, $\Big(\big((T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1}\big)P_n\Big)_{n\in\mathbb{N}}$ is strongly convergent, then

$$\sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*=\sigma_{\mathrm{ess}}\big((A_n^*)_{n\in\mathbb{N}}\big)^*.$$

Proof. i) Let $I\subset\mathbb{N}$ be an infinite subset, and let $x_n\in\mathcal{D}(T_n)$, $n\in I$, satisfy $\|x_n\|=1$ and $x_n\xrightarrow{w}0$. Lemma 2.11 ii) implies $B_nx_n\to 0$. Now the first claim follows immediately.

Now assume that, in addition, $(B_nP_n)_{n\in\mathbb{N}}$ is strongly convergent. Then, by [3, Proposition 2.10], the sequence $(B_n^*)_{n\in\mathbb{N}}$ is discretely compact. Now the second claim follows analogously as the first claim.

ii) By i), we have

$$\sigma_{\mathrm{ess}}\Big(\big((T_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}\Big)=\sigma_{\mathrm{ess}}\Big(\big((A_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}\Big).$$

Now the first claim follows from the mapping result in Theorem 2.5.

If, in addition, $\Big(\big((T_n-\lambda)^{-1}-(A_n-\lambda)^{-1}\big)P_n\Big)_{n\in\mathbb{N}}$ is strongly convergent, then [3, Proposition 2.10] implies that $\Big(\big((T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1}\big)^*P_n\Big)_{n\in\mathbb{N}}$ is discretely compact. Now the second claim follows analogously. □

2.3. **Proof of local spectral convergence result and example.** In this subsection we prove the local spectral exactness result in Theorem 2.3 and then illustrate it for the Galerkin method of perturbed Toeplitz operators.

First we establish relations of the limiting essential spectrum with the following two notions of limiting approximate point spectrum and region of boundedness (introduced by Kato [23, Section VIII.1]).

Definition 2.13. The *limiting approximate point spectrum* of $(T_n)_{n\in\mathbb{N}}$ is defined as

$$\sigma_{\mathrm{app}}\big((T_n)_{n\in\mathbb{N}}\big):=\left\{\lambda\in\mathbb{C}:\,\exists\, I\subset\mathbb{N}\,\exists\, x_n\in\mathcal{D}(T_n),\ n\in I,\text{with}\ \begin{array}{l}\|x_n\|=1,\\ \|(T_n-\lambda)x_n\|\to 0\end{array}\right\},$$

and the *region of boundedness* of $(T_n)_{n\in\mathbb{N}}$ is

$$\Delta_b\left((T_n)_{n\in\mathbb{N}}\right):=\left\{\lambda\in\mathbb{C}:\,\exists\, n_0\in\mathbb{N}\,\text{with}\ \begin{array}{l}\lambda\in\varrho(T_n),\ n\geq n_0,\\ \big(\|(T_n-\lambda)^{-1}\|\big)_{n\geq n_0}\ \text{bounded}\end{array}\right\}.$$

The following lemma follows easily from Definitions 2.2 and 2.13.

Lemma 2.14. i) *We have* $\sigma_{\text{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \subset \sigma_{\text{app}}\big((T_n)_{n\in\mathbb{N}}\big)$.

ii) *In general,*

$$\mathbb{C}\backslash\Delta_b\big((T_n)_{n\in\mathbb{N}}\big) = \sigma_{\text{app}}\big((T_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\text{app}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*.$$

If T_n, $n \in \mathbb{N}$, *all have compact resolvents, then*

$$\mathbb{C}\backslash\Delta_b\left((T_n)_{n\in\mathbb{N}}\right) = \sigma_{\text{app}}\big((T_n)_{n\in\mathbb{N}}\big) = \sigma_{\text{app}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*.$$

Under generalised strong resolvent convergence we obtain the following relations.

Proposition 2.15. i) *If* $T_n \stackrel{gsr}{\to} T$, *then*

$$\sigma_{\text{app}}(T) \subset \sigma_{\text{app}}\left((T_n)_{n\in\mathbb{N}}\right).$$

ii) *If* $T_n \stackrel{gsr}{\to} T$, *then*

$$\sigma_{\text{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \subset \sigma_{\text{app}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \subset \sigma_{\text{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \cup \sigma_p(T^*)^*.$$

iii) *If* $T_n \stackrel{gsr}{\to} T$ *and* $T_n^* \stackrel{gsr}{\to} T^*$, *then*

$$\sigma_{\text{app}}\big((T_n)_{n\in\mathbb{N}}\big) = \sigma_{\text{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \cup \sigma_p(T),$$
$$\sigma_{\text{app}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* = \sigma_{\text{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \cup \sigma_p(T^*)^*.$$

Proof. i) The proof is analogous to the proof of Proposition 2.7; the only difference is that here weak convergence of the considered elements is not required.

ii) The first inclusion follows from Lemma 2.14 i).

Let $\lambda \in \sigma_{\text{app}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*$. Then there exist an infinite subset $I \subset \mathbb{N}$ and $x_n \in \mathcal{D}(T_n^*)$, $n \in I$, with $\|x_n\| = 1$ and $\|(T_n^* - \overline{\lambda})\| \to 0$ as $n \to \infty$. Since $(x_n)_{n\in I}$ is a bounded sequence and H_0 is weakly compact, there exists $\widetilde{I} \subset I$ such that $(x_n)_{n\in\widetilde{I}}$ converges weakly to some $x \in H_0$. If $x = 0$, then $\lambda \in \sigma_{\text{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*$.

Now assume that $x \neq 0$. Since $T_n \stackrel{gsr}{\to} T$, there exists $\lambda_0 \in \Delta_b\left((T_n)_{n\in\mathbb{N}}\right) \cap \varrho(T)$ such that $(T_n - \lambda_0)^{-1}P_n \stackrel{s}{\to} (T - \lambda_0)^{-1}P$. The convergence $\|(T_n^* - \overline{\lambda})x_n\| \to 0$ implies

$$(T_n^* - \overline{\lambda_0})x_n = (\overline{\lambda} - \overline{\lambda_0})x_n + y_n, \quad \text{with} \quad y_n := (T_n^* - \overline{\lambda})x_n \longrightarrow 0, \quad n \to \infty,$$

hence

$$(T_n^* - \overline{\lambda_0})^{-1}x_n = (\overline{\lambda} - \overline{\lambda_0})^{-1}x_n - \widetilde{y}_n, \quad \widetilde{y}_n := (\overline{\lambda} - \overline{\lambda_0})^{-1}(T_n^* - \overline{\lambda_0})^{-1}y_n,$$
$$\|\widetilde{y}_n\| \leq |\lambda - \lambda_0|^{-1}\big\|(T_n - \lambda_0)^{-1}\big\|\|y_n\| \longrightarrow 0, \quad n \to \infty.$$

Since $x_n \stackrel{w}{\to} x$, we obtain $(T_n^* - \overline{\lambda_0})^{-1}x_n \stackrel{w}{\to} (\overline{\lambda} - \overline{\lambda_0})^{-1}x$ as $n \in \widetilde{I}$, $n \to \infty$. On the other hand, Lemma 2.11 i) yields $x \in H$ and $(T_n^* - \overline{\lambda_0})^{-1}x_n \stackrel{w}{\to} (T^* - \overline{\lambda_0})^{-1}x$. By the uniqueness of the weak limit, we obtain $(T^* - \overline{\lambda_0})^{-1}x = (\overline{\lambda} - \overline{\lambda_0})^{-1}x$, hence $(\overline{\lambda} - \overline{\lambda_0})^{-1} \in \sigma_p((T^* - \overline{\lambda_0})^{-1})$. This yields $\lambda \in \sigma_p(T^*)^*$.

iii) The second equality follows from claim ii), from $\sigma_p(T^*)^* \subset \sigma_{\text{app}}(T^*)^*$ and from claim i) (applied to T^*, T_n^*). Now we obtain the first equality by replacing T^*, T_n^* by T, T_n. □

The limiting essential spectrum is related to the region of boundedness as follows.

Proposition 2.16. i) *If* $T_n \stackrel{gsr}{\to} T$ *and* $T_n^* \stackrel{gsr}{\to} T^*$, *then*

$$\mathbb{C}\backslash\Delta_b\left((T_n)_{n\in\mathbb{N}}\right) = \sigma_p(T) \cup \sigma_{\text{ess}}\left((T_n)_{n\in\mathbb{N}}\right) \cup \sigma_p(T^*)^* \cup \sigma_{\text{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*,$$
$$\Delta_b\big((T_n)_{n\in\mathbb{N}}\big) \cap \varrho(T) = \big(\mathbb{C}\backslash(\sigma_{\text{ess}}\left((T_n)_{n\in\mathbb{N}}\right) \cup \sigma_{\text{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*)\big) \cap \varrho(T).$$

ii) *If* $T_n \overset{gsr}{\to} T$ *and* T_n, $n \in \mathbb{N}$, *all have compact resolvents, then*

$$\mathbb{C}\backslash\Delta_b\left((T_n)_{n\in\mathbb{N}}\right) \subset \sigma_p(T^*)^* \cup \sigma_{\mathrm{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*,$$

$$\Delta_b\big((T_n)_{n\in\mathbb{N}}\big) \cap \varrho(T) = \big(\mathbb{C}\backslash\sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*\big) \cap \varrho(T).$$

Proof. i) The claimed identities follow from Lemma 2.14 ii) and Proposition 2.15 iii).

ii) The claims follow from the second part of Lemma 2.14 ii) and Proposition 2.15 ii). □

The local spectral convergence result (Theorem 2.3) relies on the following result from [3].

Theorem 2.17. [3, Theorem 2.3] *Suppose that* $T_n \overset{gsr}{\to} T$.

i) *For each* $\lambda \in \sigma(T)$ *such that for some* $\varepsilon > 0$ *we have*

$$B_\varepsilon(\lambda)\backslash\{\lambda\} \subset \Delta_b\left((T_n)_{n\in\mathbb{N}}\right) \cap \varrho(T),$$

there exist $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, *with* $\lambda_n \to \lambda$ *as* $n \to \infty$.

ii) *No spectral pollution occurs in* $\Delta_b\left((T_n)_{n\in\mathbb{N}}\right)$.

Proof of Theorem 2.3. i) First note that the set in (2.1) is closed by Proposition 2.4. If λ is an isolated point of $\sigma(T)$ and does not belong to the set in (2.1), then there exists $\varepsilon > 0$ so small that

$$B_\varepsilon(\lambda)\backslash\{\lambda\} \subset \Big(\mathbb{C}\backslash\big(\sigma_{\mathrm{ess}}\left((T_n)_{n\in\mathbb{N}}\right) \cup \sigma_{\mathrm{ess}}\left((T_n^*)_{n\in\mathbb{N}}\right)^*\big)\Big) \cap \varrho(T).$$

By Proposition 2.16 i), the right hand side coincides with $\Delta_b\big((T_n)_{n\in\mathbb{N}}\big) \cap \varrho(T)$. Now the claims follow from Theorem 2.17.

ii) The proof is analogous to i); we use claim ii) of Proposition 2.16.

iii) Assume that the claim is false. Then there exist $\alpha > 0$, an infinite subset $I \subset \mathbb{N}$ and $\lambda_n \in K$, $n \in I$, such that one of the following holds:

(1) $\lambda_n \in \sigma(T_n)$ and $\operatorname{dist}(\lambda_n, \sigma(T) \cap K) > \alpha$ for every $n \in I$;
(2) $\lambda_n \in \sigma(T)$ and $\operatorname{dist}(\lambda_n, \sigma(T_n) \cap K) > \alpha$ for every $n \in I$.

Note that, in both cases (1) and (2), the compactness of K implies that there exist $\lambda \in K$ and an infinite subset $J \subset I$ such that $(\lambda_n)_{n\in J}$ converges to λ.

First we consider case (1). There are $\lambda_n \in \sigma(T_n)$, $n \in J$, with $\lambda_n \to \lambda \in K$. Since K does not contain spectral pollution by the assumptions, we conclude $\lambda \in \sigma(T) \cap K$. Hence

$$|\lambda_n - \lambda| \geq \operatorname{dist}(\lambda_n, \sigma(T) \cap K) > \alpha, \quad n \in J,$$

a contradiction to $\lambda_n \to \lambda$.

Now assume that (2) holds. The closedness of $\sigma(T) \cap K$ yields $\lambda \in \sigma(T) \cap K$, and the latter set is discrete by the assumptions. So there exists $n_0 \in \mathbb{N}$ so that $\lambda = \lambda_n$ for all $n \in J$ with $n \geq n_0$. In addition, by the above claim i) or ii), respectively, there exist $\mu_n \in \sigma(T_n)$, $n \in \mathbb{N}$, so that $\mu_n \to \lambda$ as $n \to \infty$. Since $\lambda \in \sigma(T) \cap K$ is in the interior of K by the assumptions, there exists $n_1 \in \mathbb{N}$ so that $\mu_n \in K$ for all $n \geq n_1$. So we conclude that, for all $n \in J$ with $n \geq \max\{n_0, n_1\}$,

$$|\lambda - \mu_n| \geq \operatorname{dist}(\lambda, \sigma(T_n) \cap K)) = \operatorname{dist}(\lambda_n, \sigma(T_n) \cap K) > \alpha,$$

a contradiction to $\mu_n \to \lambda$. This proves the claim. □

It is well known that truncating a Toeplitz operator (and compact perturbations of it) to finite sections is not a spectrally exact process but the pseudospectra converge in Hausdorff metric, see [5, 30] and [6, Theorem 3.17, Corollary 3.18 (b)] (where non-strict inequality in the definition of pseudospectra is used). In the following example we illustrate Theorem 2.3 for the Galerkin method of a compact perturbation of a Toeplitz operator using the perturbation result for the limiting essential spectrum (Theorem 2.12).

Example 2.18. Denote by $\{e_k : k \in \mathbb{N}\}$ the standard orthonormal basis of $l^2(\mathbb{N})$. Let $T \in L(l^2(\mathbb{N}))$ be the *Toeplitz operator* defined by the so-called *symbol*

$$f(z) := \sum_{k\in\mathbb{Z}} a_k z^k, \quad z \in \mathbb{C},$$

where $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$, are chosen so that f is continuous. This means that, with respect to $\{e_k : k \in \mathbb{N}\}$, the operator T has the matrix representation $(T_{ij})_{i,j=1}^{\infty}$ with

$$T_{ij} := \langle Te_j, e_i\rangle = a_{i-j}, \quad i, j \in \mathbb{N}.$$

The set $f(\partial B_1(0))$ is called *symbol curve.* Given $\lambda \notin f(\partial B_1(0))$, we define the *winding number* $I(f, \lambda)$ to be the winding number of $f(\partial B_1(0))$ about λ in the usual positive (counterclockwise) sense. The spectrum of T is, by [6, Theorem 1.17], given by

$$\sigma(T) = f(\partial B_1(0)) \cup \big\{\lambda \notin f(\partial B_1(0)) : I(f, \lambda) \neq 0\big\}.$$

For $n \in \mathbb{N}$, let P_n be the orthogonal projection of $l^2(\mathbb{N})$ onto $H_n := \operatorname{span}\{e_k : k = 1, \dots, n\}$. It is easy to see that $P_n \xrightarrow{s} I$. For a compact operator $S \in L(l^2(\mathbb{N}))$, let $A := T + S$ and define $A_n := P_n A|_{H_n}$, $n \in \mathbb{N}$. We claim that the limiting essential spectra satisfy

$$\sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((A_n^*)_{n\in\mathbb{N}}\big)^* \subset \sigma(T) \subset \sigma(A); \tag{2.8}$$

hence, by Theorem 2.3, no spectral pollution occurs for the approximation $(A_n)_{n\in\mathbb{N}}$ of A, and every isolated $\lambda \in \sigma(A)\backslash\sigma(T)$ is the limit of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \sigma(A_n)$, $n \in \mathbb{N}$.

To prove these statements, define $T_n := P_n T|_{H_n}$, $n \in \mathbb{N}$. Clearly, $T_n P_n \xrightarrow{s} T$, $A_n P_n \xrightarrow{s} A$ and $T_n^* P_n \xrightarrow{s} T^*$, $A_n^* P_n \xrightarrow{s} A^*$. Hence $T_n \xrightarrow{gsr} T$, $A_n \xrightarrow{gsr} A$ and $T_n^* \xrightarrow{gsr} T^*$, $A_n^* \xrightarrow{gsr} A^*$. By [6, Theorem 2.11], $\varrho(T) \subset \Delta_b\left((T_n)_{n\in\mathbb{N}}\right)$. Using Proposition 2.16 i), we obtain

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \subset \mathbb{C}\backslash\Delta_b\big((T_n)_{n\in\mathbb{N}}\big).$$

The perturbation result in Theorem 2.12 i) implies

$$\sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((A_n^*)_{n\in\mathbb{N}}\big)^* = \sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*.$$

So, altogether we arrive at the first inclusion in (2.8). By [6, Theorem 1.17], $\sigma_{\mathrm{ess}}(T) \cup \sigma_{\mathrm{ess}}(T^*)^* = f(\partial B_1(0))$ and for $\lambda \in \sigma(T)\backslash f(\partial B_1(0))$ the operator $T - \lambda$ is Fredholm with index $\operatorname{ind}(T - \lambda) = -I(f, \lambda) \neq 0$. This means that $\sigma(T)$ is equal to the set $\sigma_{e4}(T)$ defined in [16, Chapter IX], one of the (in general not equivalent) characterisations of essential spectrum. This set is invariant under compact perturbations by [16, Theorem IX.2.1], hence $\sigma_{e4}(T) = \sigma_{e4}(A) \subset \sigma(A)$, which proves the second inclusion in (2.8). The rest of the claim follows from Theorem 2.3.

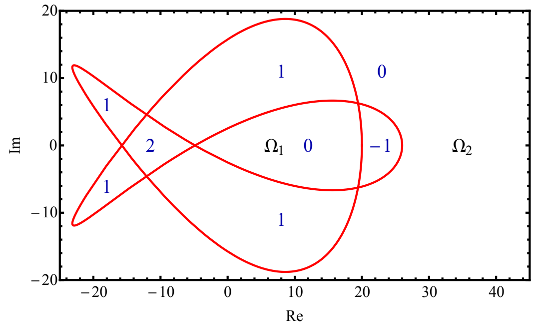
For a concrete example, let

$$\begin{aligned} a_{-3} = -7, \quad a_{-2} = 8, \quad a_{-1} = -1, \quad a_2 = 15, \quad a_3 = 5,\\ a_k = 0, \quad k \in \mathbb{Z}\backslash\{-3, -2, -1, 2, 3\}. \end{aligned}$$

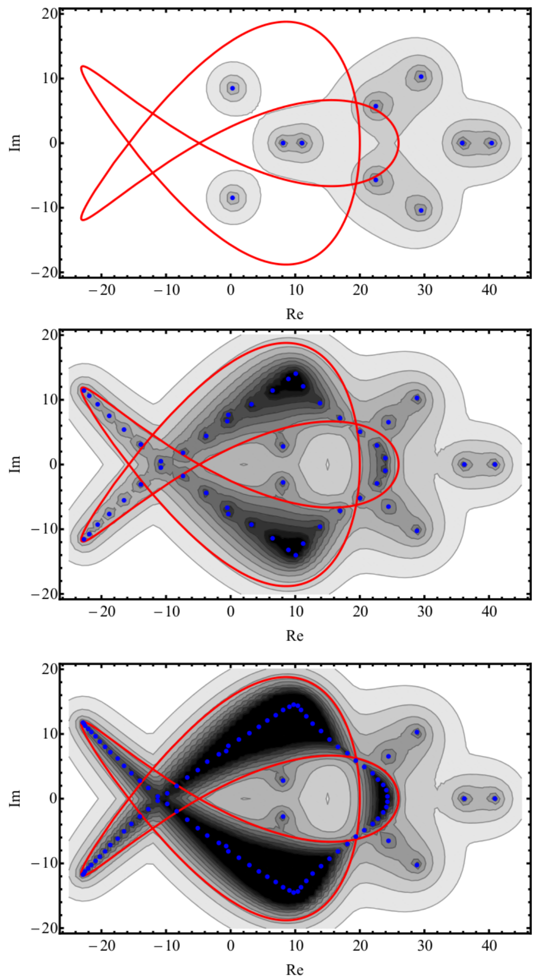
The symbol curve $f(\partial B_1(0))$ is shown in Figure 1 in red. The spectrum of the corresponding Toeplitz operator T consists of the symbol curve together with the connected components with winding numbers $1, 2, -1$, see Figure 1 (a). So the resolvent set $\varrho(T)$ is the union of the connected components with winding number 0, which are denoted by Ω_1 and Ω_2 in Figure 1 (a).

Now we add the compact operator S with matrix representation

$$(S_{ij})_{i,j=1}^{\infty}, \quad S_{ij} := \begin{cases} 20, & i = j \leq 10,\\ 0, & \text{otherwise.} \end{cases}$$



(a) Symbol curve (red) corresponding to T and winding number in each component.



(b) Eigenvalues (blue dots) of A_n for $n = 10$ (top), $n = 50$ (middle), $n = 100$ (bottom) and ε-pseudospectra of A_n for $\varepsilon = 2, 1, 2^{-1}, \dots, 2^{-5}$.

FIGURE 1. Spectra and pseudospectra of the truncated $n \times n$ matrices A_n of the perturbed Toeplitz operator A.

By the above claim, the Galerkin approximation $(A_n)_{n\in\mathbb{N}}$ of $A := T + S$ does not produce spectral pollution, and every accumulation point of $\sigma(A_n)$, $n \in \mathbb{N}$, in

$\Omega_1 \cup \Omega_2$ belongs to $\sigma(A)$. Figure 1 (b) suggests that two such accumulation points exist in Ω_1 and six in Ω_2.

Note that although the points in $\sigma(T) \subset \sigma(A)$ are not approximated by the Galerkin method, the resolvent norm diverges at these points, see Figure 1 (b). This is justified by [7, Proposition 4.2], which implies that for every $\lambda \in \sigma(A)$ and every $\varepsilon > 0$ there exists $n_{\lambda,\varepsilon} \in \mathbb{N}$ with $\lambda \in \sigma_\varepsilon(A_n)$, $n \geq n_{\lambda,\varepsilon}$ (see also Theorem 3.3 below). Moreover, for any $\varepsilon > 0$, in the limit $n \to \infty$ the closed ε-pseudospectrum $\overline{\sigma_\varepsilon(A_n)}$ converges to $\overline{\sigma_\varepsilon(A)} \cup \overline{\sigma_\varepsilon(T)}$ with respect to the Hausdorff metric; this follows from [6, Corollary 3.18 (b)] and since every bounded Hilbert space operator B satisfies $\overline{\sigma_\varepsilon(B)} = \{\lambda \in \mathbb{C} : \|(B-\lambda)^{-1}\| \geq 1/\varepsilon\}$ by e.g. [5, Proposition 6.1].

3. Local convergence of pseudospectra

In this section we establish special properties and convergence of pseudospectra. Subsection 3.1 contains the main pseudospectral convergence result (Theorem 3.6). We also study the special case of operators having constant resolvent norm on an open set (Theorem 3.8). In Subsection 3.2 we provide properties of the limiting essential ε-near spectrum $\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$ including a perturbation result (Theorem 3.15), followed by Subsection 3.3 with the proofs of the results stated in Subsection 3.1.

3.1. Main convergence result.

We fix an $\varepsilon > 0$.

Definition 3.1. Define the *ε-approximate point spectrum* of T by

$$\sigma_{\mathrm{app},\varepsilon}(T) := \big\{\lambda \in \mathbb{C} : \exists\, x \in \mathcal{D}(T) \text{ with } \|x\| = 1,\ \|(T-\lambda)x\| < \varepsilon\big\}.$$

The following properties are well-known, see for instance [35, Chapter 4] and [12].

Lemma 3.2. i) *The sets $\sigma_\varepsilon(T)$, $\sigma_{\mathrm{app},\varepsilon}(T)$ are open subsets of $\mathbb{C}$.*

ii) *We have*

$$\sigma_\varepsilon(T) = \sigma_{\mathrm{app},\varepsilon}(T) \cup \sigma(T) = \sigma_{\mathrm{app},\varepsilon}(T) \cup \sigma_{\mathrm{app},\varepsilon}(T^*)^*,$$

and

$$\sigma_{\mathrm{app},\varepsilon}(T) \backslash \sigma(T) = \sigma_{\mathrm{app},\varepsilon}(T^*)^* \backslash \sigma(T).$$

If T has compact resolvent, then

$$\sigma_\varepsilon(T) = \sigma_{\mathrm{app},\varepsilon}(T) = \sigma_{\mathrm{app},\varepsilon}(T^*)^*.$$

iii) *For $\varepsilon > \varepsilon' > 0$,*

$$\sigma_{\varepsilon'}(T) \subset \sigma_\varepsilon(T), \qquad \bigcap_{\varepsilon>0} \sigma_\varepsilon(T) = \sigma(T).$$

iv) *We have*

$$\big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(T)\big) < \varepsilon\big\} \subset \sigma_\varepsilon(T),$$

with equality if T is selfadjoint.

In contrast to the spectrum, the ε-pseudospectrum is always approximated under generalised strong resolvent convergence. For bounded operators and strong convergence, this was proved by Böttcher-Wolf in [7, Proposition 4.2] (where non-strict inequality in the definition of pseudospectra is used); claim i) is not explicitly stated but can be read off from the proof. Note that if T has compact resolvent, then claim i) holds for all $\lambda \in \sigma_\varepsilon(T) = \sigma_{\mathrm{app},\varepsilon}(T)$ by Lemma 3.2 ii).

Theorem 3.3. *Suppose that $T_n \stackrel{gsr}{\to} T$.*

i) *For every* $\lambda \in \sigma_{\mathrm{app},\varepsilon}(T)$ *and* $x \in \mathcal{D}(T)$ *with* $\|x\| = 1$, $\|(T-\lambda)x\| < \varepsilon$ *there exist* $n_\lambda \in \mathbb{N}$ *and* $x_n \in \mathcal{D}(T_n)$, $n \geq n_\lambda$, *with*

$$\lambda \in \sigma_{\mathrm{app},\varepsilon}(T_n), \quad \|x_n\| = 1, \quad \|(T_n - \lambda)x_n\| < \varepsilon, \quad n \geq n_\lambda,$$

and $\|x_n - x\| \to 0$ *as* $n \to \infty$.

ii) *Suppose that, in addition,* $T_n^* \overset{gsr}{\to} T^*$. *Then for every* $\lambda \in \sigma_\varepsilon(T)$ *there exists* $n_\lambda \in \mathbb{N}$ *such that* $\lambda \in \sigma_\varepsilon(T_n)$, $n \geq n_\lambda$.

The following example illustrates that we cannot omit the additional assumption $T_n^* \overset{gsr}{\to} T^*$ in Theorem 3.3 ii). In particular, this is a counterexample for [9, Theorem 4.4] where only $T_n \overset{gsr}{\to} T$ is assumed.

Example 3.4. Let T be the first derivative in $L^2(0,\infty)$ with Dirichlet boundary condition,

$$Tf := f', \quad \mathcal{D}(T) := \{f \in W^{1,2}(0,\infty) : f(0) = 0\}.$$

We approximate T by a sequence of operators T_n in $L^2(0,n)$, $n \in \mathbb{N}$, defined by

$$T_n f := f', \quad \mathcal{D}(T_n) := \{f \in W^{1,2}(0,n) : f(0) = f(n)\}.$$

Note that $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\} = \sigma(T) \subset \sigma_\varepsilon(T)$. The operators $\mathrm{i}\,T_n$, $n \in \mathbb{N}$, are selfadjoint. Hence

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\} \subset \Delta_b\big((T_n)_{n\in\mathbb{N}}\big) \cap \varrho(T).$$

Using that $\{f \in \mathcal{D}(T) : \operatorname{supp} f \text{ compact}\}$ is a core of T, [3, Theorem 3.1] implies that $T_n \overset{gsr}{\to} T$. However, since $\mathrm{i}\,T$ is not selfadjoint, we obtain $T_n^* = -T_n \overset{gsr}{\to} -T \neq T^*$. The selfadjointness of $\mathrm{i}\,T_n$ and Lemma 3.2 iv) imply

$$\sigma_\varepsilon(T_n) = \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(T_n)) < \varepsilon\} \subset \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| < \varepsilon\}, \quad n \in \mathbb{N}.$$

Therefore, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \varepsilon$, we conclude $\lambda \in \sigma_\varepsilon(T)$ but

$$\operatorname{dist}(\lambda, \sigma_\varepsilon(T_n)) \geq \operatorname{Re} \lambda - \varepsilon > 0, \quad n \in \mathbb{N}.$$

In order to characterise ε-pseudospectral pollution, we introduce the following sets.

Definition 3.5. Define the *essential* ε*-near spectrum* of T by

$$\Lambda_{\mathrm{ess},\varepsilon}(T) := \left\{\lambda \in \mathbb{C} : \exists\, x_n \in \mathcal{D}(T),\, n \in \mathbb{N}, \text{ with } \begin{array}{l} \|x_n\| = 1,\, x_n \overset{w}{\to} 0, \\ \|(T-\lambda)x_n\| \to \varepsilon \end{array}\right\},$$

and the *limiting essential* ε*-near spectrum* of $(T_n)_{n\in\mathbb{N}}$ by

$$\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) := \left\{\lambda \in \mathbb{C} : \begin{array}{l} \exists\, I \subset \mathbb{N}\, \exists\, x_n \in \mathcal{D}(T_n),\, n \in I, \text{ with} \\ \|x_n\| = 1,\, x_n \overset{w}{\to} 0,\, \|(T_n - \lambda)x_n\| \to \varepsilon \end{array}\right\}.$$

The following theorem is the main result of this section. We establish local ε-pseudospectral exactness and prove ε-pseudospectral convergence with respect to the Hausdorff metric in compact subsets of the complex plane where we have ε-pseudospectral exactness.

Theorem 3.6. *Suppose that* $T_n \overset{gsr}{\to} T$ *and* $T_n^* \overset{gsr}{\to} T^*$.

i) *The sequence* $(T_n)_{n\in\mathbb{N}}$ *is an* ε*-pseudospectrally inclusive approximation of* T.

ii) *Define*

$$\Lambda_{\mathrm{ess},(0,\varepsilon]} := \bigcup_{\delta\in(0,\varepsilon]} \Big(\Lambda_{\mathrm{ess},\delta}\big((T_n)_{n\in\mathbb{N}}\big) \cap \Lambda_{\mathrm{ess},\delta}\big((T_n^*)_{n\in\mathbb{N}}\big)^*\Big).$$

Then ε*-pseudospectral pollution is confined to*

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \cup \Lambda_{\mathrm{ess},(0,\varepsilon]}; \tag{3.1}$$

if the operators T_n, $n \in \mathbb{N}$, all have compact resolvents, then it is restricted to

$$\big(\sigma_{\rm ess}\big((T_n)_{n\in\mathbb{N}}\big) \cap \sigma_{\rm ess}\big((T_n^*)_{n\in\mathbb{N}}\big)^*\big) \cup \Lambda_{{\rm ess},(0,\varepsilon]}. \tag{3.2}$$

iii) *Let $K \subset \mathbb{C}$ be a compact subset with*

$$\overline{\sigma_\varepsilon(T)} \cap K = \overline{\sigma_\varepsilon(T) \cap K} \neq \emptyset.$$

If the intersection of K with the set in (3.1) *or* (3.2)*, respectively, is contained in $\overline{\sigma_\varepsilon(T)}$, then*

$$d_{\rm H}\big(\overline{\sigma_\varepsilon(T_n)} \cap K, \overline{\sigma_\varepsilon(T)} \cap K\big) \longrightarrow 0, \quad n \to \infty.$$

Remark 3.7. If we compare Theorem 3.6 iii) with [4, Theorem 2.1] for generalised *norm* resolvent convergence, note that here we do no explicitly exclude the possibility that $\lambda \mapsto \|(T-\lambda)^{-1}\|$ is constant on an open subset $\emptyset \neq U \subset \varrho(T)$. However, if the resolvent norm is equal to $1/\varepsilon$ on an open set U, then $U \cap \overline{\sigma_\varepsilon(T)} = \emptyset$ and hence the following Theorem 3.8 ii) implies that a compact set K with $K \cap \Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) \subset \overline{\sigma_\varepsilon(T)}$ satisfies $K \cap U = \emptyset$. So we implicitly exclude the problematic region U.

In the following result we study operators that have constant resolvent norm on an open set. For the existence of such operators see [31, 4].

Theorem 3.8. *Assume that there exists an open subset $\emptyset \neq U \subset \varrho(T)$ such that*

$$\|(T-\lambda)^{-1}\| = \frac{1}{\varepsilon}, \quad \lambda \in U.$$

i) *We have*

$$\varrho(T) \subset \mathbb{C}\backslash \bigcap_{K \text{ compact}} \sigma(T+K) \subset \Lambda_{{\rm ess},\varepsilon}(T) \cap \Lambda_{{\rm ess},\varepsilon}(T^*)^*.$$

ii) *If $T_n \stackrel{gsr}{\to} T$ and $T_n^* \stackrel{gsr}{\to} T^*$, then*

$$\varrho(T) \subset \mathbb{C}\backslash \bigcap_{K \text{ compact}} \sigma(T+K) \subset \Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) \cap \Lambda_{{\rm ess},\varepsilon}\big((T_n^*)_{n\in\mathbb{N}}\big)^*.$$

Remark 3.9. Note that, by [16, Section IX.1, Theorems IX.1.3, 1.4],

$$\sigma_{\rm ess}(T) \cup \sigma_{\rm ess}(T^*)^* = \sigma_{e3}(T) \subset \sigma_{e4}(T) = \bigcap_{K \text{ compact}} \sigma(T+K).$$

3.2. Properties of the limiting essential ε-near spectrum.

Proposition 3.10. i) *The sets $\Lambda_{{\rm ess},\varepsilon}(T)$, $\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$ are closed subsets of $\mathbb{C}$.*

ii) *We have*

$$\begin{aligned} \big\{\lambda + z : \lambda \in \sigma_{\rm ess}(T),\, |z| = \varepsilon\big\} &\subset \Lambda_{{\rm ess},\varepsilon}(T), \\ \big\{\lambda + z : \lambda \in \sigma_{\rm ess}\big((T_n)_{n\in\mathbb{N}}\big),\, |z| = \varepsilon\big\} &\subset \Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big). \end{aligned} \tag{3.3}$$

Proof. A diagonal sequence argument implies claim i), and claim ii) is easy to see. □

Remark 3.11. The inclusions in claim ii) may be strict. In fact, for Shargorodsky's example [31, Theorem 3.2] of an operator T with constant ($1/\varepsilon = 1$) resolvent norm on an open set, the compressions T_n onto the span of the first $2n$ basis vectors satisfy

$$\sigma_{\rm ess}\big((T_n)_{n\in\mathbb{N}}\big) = \sigma_{\rm ess}(T) = \emptyset, \qquad \bigcap_{K \text{ compact}} \sigma(T+K) = \emptyset.$$

Hence the left hand side of (3.3) is empty whereas the right hand side equals $\mathbb{C}$ by Theorem 3.8.

Analogously as $\sigma_{\rm ess}(T)\subset\sigma_{\rm ess}\big((T_n)_{n\in\mathbb{N}}\big)$ (see Proposition 2.7), also the essential ε-near spectrum is contained in its limiting counterpart.

Proposition 3.12. i) *Assume that* $T_n\stackrel{gsr}{\to}T$. *Then* $\Lambda_{{\rm ess},\varepsilon}(T)\subset\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$.
ii) *If* $H_0=H$ *and* $T_n\stackrel{gnr}{\to}T$, *then* $\Lambda_{{\rm ess},\varepsilon}(T)\cap\varrho(T)=\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)\cap\varrho(T)$.

For the proof we use the following simple result.

Lemma 3.13. *Assume that* $T_n^*\stackrel{gsr}{\to}T^*$. *Suppose that there exist an infinite subset* $I\subset\mathbb{N}$ *and* $x_n\in\mathcal{D}(T_n)$, $n\in I$, *with* $\|x_n\|=1$ *and* $x_n\stackrel{w}{\to}0$. *If* $(\|T_nx_n\|)_{n\in I}$ *is bounded, then* $T_nx_n\stackrel{w}{\to}0$.

Proof. Define $y_n:=T_nx_n$. Let $\lambda_0\in\Delta_b\big((T_n)_{n\in\mathbb{N}}\cap\varrho(T)$ satisfy $(T_n^*-\overline{\lambda_0})^{-1}P_n\stackrel{s}{\to}(T^*-\overline{\lambda_0})^{-1}P$. Since H_0 is weakly compact, there exist $y\in H_0$ and an infinite subset $\widetilde{I}\subset I$ such that $(y_n)_{n\in\widetilde{I}}$ converges weakly to y. We prove that $y=0$. Lemma 2.11 i) implies $y\in H$ and $(T_n-\lambda_0)^{-1}y_n\stackrel{w}{\to}(T-\lambda_0)^{-1}y$. Hence $x_n=(T_n-\lambda_0)^{-1}y_n-\lambda_0x_n\stackrel{w}{\to}(T-\lambda_0)^{-1}y$. The uniqueness of the weak limit yields $(T-\lambda_0)^{-1}y=0$ and thus $y=0$. □

Proof of Proposition 3.12. i) The proof is analogous to the one of Proposition 2.7 i).
ii) Using claim i), it remains to prove $\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)\cap\varrho(T)\subset\Lambda_{{\rm ess},\varepsilon}(T)$. Let $\lambda\in\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)\cap\varrho(T)$. By Definition 3.5, there exist an infinite subset $I\subset\mathbb{N}$ and $x_n\in\mathcal{D}(T_n)$, $n\in I$, with $\|x_n\|=1$, $x_n\stackrel{w}{\to}0$ and $\|(T_n-\lambda)x_n\|\to\varepsilon$. Since $\lambda\in\varrho(T)$, [3, Proposition 2.16 ii)] implies that there exists $n_\lambda\in\mathbb{N}$ such that $\lambda\in\varrho(T_n)$, $n\geq n_\lambda$, and $(T_n-\lambda)^{-1}P_n\to(T-\lambda)^{-1}$. Define $I_2:=\{\lambda\in I:\,n\geq n_\lambda\}$ and

$$w_n:=\frac{(T_n-\lambda)x_n}{\|(T_n-\lambda)x_n\|}\in H_n\subset H,\quad n\in I_2.$$

Then $\|w_n\|=1$ and $w_n\stackrel{w}{\to}0$ by Lemma 3.13. In addition,

$$\|(T_n-\lambda)^{-1}w_n\|=\frac{1}{\|(T_n-\lambda)x_n\|}\longrightarrow\frac{1}{\varepsilon},\quad n\in I_2,\quad n\to\infty.$$

Since, in the limit $n\in I_2$, $n\to\infty$,

$$\big|\|(T_n-\lambda)^{-1}w_n\|-\|(T-\lambda)^{-1}w_n\|\big|\leq\|(T_n-\lambda)^{-1}P_n-(T-\lambda)^{-1}\|\longrightarrow 0,$$

we conclude $\|(T-\lambda)^{-1}w_n\|\to1/\varepsilon$. Now define

$$v_n:=\frac{(T-\lambda)^{-1}w_n}{\|(T-\lambda)^{-1}w_n\|}\in\mathcal{D}(T),\quad n\in I_2.$$

Then $\|v_n\|=1$ and $v_n\stackrel{w}{\to}0$. Moreover, $\|(T-\lambda)v_n\|=\|(T-\lambda)^{-1}w_n\|^{-1}\to\varepsilon$, hence $\lambda\in\Lambda_{{\rm ess},\varepsilon}(T)$. □

Similarly as for $\sigma_{\rm ess}\big((T_n)_{n\in\mathbb{N}}\big)$ (see Proposition 2.10), the set $\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$ is particularly simple if the operators T_n, $n\in\mathbb{N}$, or their resolvents form a discretely compact sequence.

Proposition 3.14. i) *If* $T_n\in L(H_n)$, $n\in\mathbb{N}$, *are so that* $(T_n)_{n\in\mathbb{N}}$ *is a discretely compact sequence and* $(T_n^*P_n)_{n\in\mathbb{N}}$ *is strongly convergent, then*

$$\Lambda_{{\rm ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)=\{\lambda\in\mathbb{C}:\,|\lambda|=\varepsilon\}.$$

If, in addition, $(T_nP_n)_{n\in\mathbb{N}}$ *is strongly convergent, then*

$$\Lambda_{{\rm ess},\varepsilon}\big((T_n^*)_{n\in\mathbb{N}}\big)^*=\{\lambda\in\mathbb{C}:\,|\lambda|=\varepsilon\}.$$

ii) *If there exists* $\lambda_0 \in \bigcap_{n\in\mathbb{N}} \varrho(T_n)\cap\varrho(T)$ *such that* $((T_n-\lambda_0)^{-1})_{n\in\mathbb{N}}$ *is a discretely compact sequence and* $(T_n^*-\overline{\lambda_0})^{-1}P_n \xrightarrow{s} (T^*-\overline{\lambda_0})^{-1}P$*, then*

$$\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) = \emptyset.$$

If, in addition, $(T_n-\lambda_0)^{-1}P_n \xrightarrow{s} (T-\lambda_0)^{-1}P$*, then*

$$\Lambda_{\mathrm{ess},\varepsilon}\big((T_n^*)_{n\in\mathbb{N}}\big)^* = \emptyset.$$

Proof. i) The proof is similar to the one of Proposition 2.10 i).

ii) Assume that there exists $\lambda \in \Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$. Then there are an infinite subset $I \subset \mathbb{N}$ and $x_n \in \mathcal{D}(T_n)$, $n \in I$, with

$$\|x_n\| = 1, \quad x_n \xrightarrow{w} 0, \quad \|(T_n-\lambda)x_n\| \longrightarrow \varepsilon, \quad n\in I, \quad n\to\infty.$$

Define

$$y_n := (T_n-\lambda)x_n, \quad n \in I.$$

By Lemma 3.13, $y_n \xrightarrow{w} 0$ as $n \in I$, $n\to\infty$. Hence $(T_n-\lambda_0)x_n = y_n+(\lambda-\lambda_0)x_n \xrightarrow{w} 0$ and thus, by the assumptions and Lemma 2.11 ii),

$$x_n = (T_n-\lambda_0)^{-1}\big(y_n+(\lambda-\lambda_0)x_n\big) \longrightarrow 0, \quad n\in I, \quad n\to\infty.$$

The obtained contradiction to $\|x_n\| = 1$, $n \in I$, proves the first claim.

The second claim is obtained analogously, using that $((T_n^*-\overline{\lambda_0})^{-1})_{n\in\mathbb{N}}$ is discretely compact by [3, Proposition 2.10]. □

We prove a perturbation result for $\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$ and, in claim ii), also for $\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big)$; for the latter we use that the assumptions used here imply the assumptions of Theorem 2.12.

Theorem 3.15. i) *Let* $B_n \in L(H_n)$, $n\in\mathbb{N}$. *If the sequence* $(B_n)_{n\in\mathbb{N}}$ *is discretely compact and* $(B_n^*P_n)_{n\in\mathbb{N}}$ *is strongly convergent, then*

$$\Lambda_{\mathrm{ess},\varepsilon}\left((T_n+B_n)_{n\in\mathbb{N}}\right) = \Lambda_{\mathrm{ess},\varepsilon}\left((T_n)_{n\in\mathbb{N}}\right).$$

If, in addition, $(B_nP_n)_{n\in\mathbb{N}}$ *is strongly convergent, then*

$$\Lambda_{\mathrm{ess},\varepsilon}\left(((T_n+B_n)^*)_{n\in\mathbb{N}}\right)^* = \Lambda_{\mathrm{ess},\varepsilon}\left((T_n^*)_{n\in\mathbb{N}}\right)^*.$$

ii) *Let* S *and* S_n, $n\in\mathbb{N}$, *be linear operators in* H *and* H_n, $n\in\mathbb{N}$, *with* $\mathcal{D}(T)\subset\mathcal{D}(S)$ *and* $\mathcal{D}(T_n)\subset\mathcal{D}(S_n)$, $n\in\mathbb{N}$, *respectively. Assume that there exist* $\lambda_0 \in \bigcap_{n\in\mathbb{N}} \varrho(T_n)\cap\varrho(T)$ *and* $\gamma_{\lambda_0} < 1$ *such that*

(a) $\|S(T-\lambda_0)^{-1}\| < 1$ *and* $\|S_n(T_n-\lambda_0)^{-1}\| \le \gamma_{\lambda_0}$ *for all* $n\in\mathbb{N}$*;*
(b) *the sequence* $\big(S_n(T_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}$ *is discretely compact;*
(c) *we have*

$$\left.\begin{array}{rl} (T_n^*-\overline{\lambda_0})^{-1}P_n &\xrightarrow{s} (T^*-\overline{\lambda_0})^{-1}P, \\ (S_n(T_n-\lambda_0)^{-1})^*P_n &\xrightarrow{s} (S(T-\lambda_0)^{-1})^*P, \end{array}\right. \quad n\to\infty.$$

Then the sums $A := T+S$ *and* $A_n := T_n+S_n$, $n\in\mathbb{N}$, *satisfy* $(A_n^*-\overline{\lambda_0})^{-1}P_n \xrightarrow{s} (A^*-\overline{\lambda_0})^{-1}P$ *and*

$$\Lambda_{\mathrm{ess},\varepsilon}\big((A_n)_{n\in\mathbb{N}}\big) = \Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big), \quad \sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big) = \sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big). \tag{3.4}$$

iii) *Let* S *be a linear operator in* H *with* $\mathcal{D}(T)\subset\mathcal{D}(S)$. *If there exists* $\lambda_0\in\varrho(T)$ *such that* $\|S(T-\lambda_0)^{-1}\| < 1$ *and* $S(T-\lambda_0)^{-1}$ *is compact, then*

$$\Lambda_{\mathrm{ess},\varepsilon}(T+S) = \Lambda_{\mathrm{ess},\varepsilon}(T).$$

Proof. i) The proof is analogous to the proof of Theorem 2.12 i).

ii) The proof relies on the following claim, which we prove at the end.

Claim: We have $\lambda_0 \in \bigcap_{n\in\mathbb{N}} \varrho(A_n) \cap \varrho(A)$, the sequences

$$\big((T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}, \quad \big(S_n(A_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}} \tag{3.5}$$

are discretely compact and, in the limit $n\to\infty$,

$$\begin{aligned}(A_n^*-\overline{\lambda_0})^{-1}P_n &\xrightarrow{s} (A^*-\overline{\lambda_0})^{-1}P,\\ ((T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1})^*P_n &\xrightarrow{s} ((T-\lambda_0)^{-1}-(A-\lambda_0)^{-1})^*P,\\ (S_n(A_n-\lambda_0)^{-1})^*P_n &\xrightarrow{s} (S(A-\lambda_0)^{-1})^*P.\end{aligned} \tag{3.6}$$

The second equality in (3.4) follows from the above Claim and Theorem 2.12 ii).

Now let $\lambda \in \Lambda_{\mathrm{ess},\varepsilon}\big((A_n)_{n\in\mathbb{N}}\big)$. Then there exist an infinite subset $I\subset\mathbb{N}$ and $x_n\in\mathcal{D}(A_n)$, $n\in I$, with

$$\|x_n\|=1, \quad x_n \xrightarrow{w} 0, \quad \|(A_n-\lambda)x_n\| \longrightarrow \varepsilon, \quad n\in I, \quad n\to\infty.$$

Define

$$y_n := (A_n-\lambda)x_n, \quad n\in I.$$

By Lemma 3.13, we conclude $y_n \overset{w}{\to} 0$ as $n\in I$, $n\to\infty$. Then $(A_n-\lambda_0)x_n = y_n+(\lambda-\lambda_0)x_n \overset{w}{\to} 0$ and thus, by the above Claim and Lemma 2.11 ii),

$$S_nx_n = S_n(A_n-\lambda_0)^{-1}\big(y_n+(\lambda-\lambda_0)x_n\big) \longrightarrow 0, \quad n\in I, \quad n\to\infty.$$

Therefore, $\|(T_n-\lambda)x_n\| \le \|(A_n-\lambda)x_n\|+\|S_nx_n\| \to \varepsilon$ and hence $\lambda\in\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big)$.

The reverse inclusion $\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) \subset \Lambda_{\mathrm{ess},\varepsilon}\big((A_n)_{n\in\mathbb{N}}\big)$ is proved analogously, using that $\big(S_n(T_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}$ is discretely compact and $(S_n(T_n-\lambda_0)^{-1})^*P_n \overset{s}{\to} (S(T-\lambda_0)^{-1})^*P$ by assumptions (b), (c).

Proof of Claim: A Neumann series argument implies that, for every $n\in\mathbb{N}$, we have $\lambda_0\in\varrho(A_n)$ and

$$\begin{aligned}(A_n-\lambda_0)^{-1} &= (T_n-\lambda_0)^{-1}(I+S_n(T_n-\lambda_0)^{-1})^{-1},\\ (A_n^*-\overline{\lambda_0})^{-1} &= \big(I+(S_n(T_n-\lambda_0)^{-1})^*\big)^{-1}(T_n^*-\overline{\lambda_0})^{-1},\\ (S_n(A_n-\lambda_0)^{-1})^* &= \big(I+(S_n(T_n-\lambda_0)^{-1})^*\big)^{-1}(S_n(T_n-\lambda_0)^{-1})^*;\end{aligned} \tag{3.7}$$

for A, S, T we obtain analogous equalities. Now we apply [3, Lemma 3.2] to $B=(S(T-\lambda_0)^{-1})^*$ and $B_n=(S_n(T_n-\lambda_0)^{-1})^*$, $n\in\mathbb{N}$; note that $-1\in\Delta_b\big((B_n)_{n\in\mathbb{N}}\big)\cap\varrho(B)$ by assumption (a). Hence we obtain

$$\big(I+(S_n(T_n-\lambda_0)^{-1})^*\big)^{-1}P_n \xrightarrow{s} \big(I+(S(T-\lambda_0)^{-1})^*\big)^{-1}P.$$

Now the strong convergences (3.6) follow from (3.7) and assumption (c). To prove discrete compactness of the sequences in (3.5), we use that

$$\begin{aligned}(T_n-\lambda_0)^{-1}-(A_n-\lambda_0)^{-1} &= (A_n-\lambda_0)^{-1}S_n(T_n-\lambda_0)^{-1},\\ S_n(A_n-\lambda_0)^{-1} &= S_n(T_n-\lambda_0)^{-1}(I+S_n(T_n-\lambda_0)^{-1})^{-1}.\end{aligned}$$

Now the claims are obtained by [3, Lemma 2.8 i), ii)] and using assumptions (a), (b) and $(A_n^*-\overline{\lambda_0})^{-1}P_n \overset{s}{\to} (A^*-\overline{\lambda_0})^{-1}P$ by (3.6).

iii) The assertion follows from claim ii) applied to $T_n=T$, $S_n=S$, $n\in\mathbb{N}$. $\square$

3.3. **Proofs of pseudospectral convergence results.** First we prove the ε-pseudospectral inclusion result.

Proof of Theorem 3.3. i) The assumption $T_n \stackrel{gsr}{\to} T$ and Lemma 2.8 imply that there are $x_n \in \mathcal{D}(T_n)$, $n \in \mathbb{N}$, with $\|x_n\| = 1$, $\|x_n - x\| \to 0$, $\|T_n x_n - Tx\| \to 0$. Hence there exists $n_\lambda \in \mathbb{N}$ such that, for all $n \geq n_\lambda$,

$$\|(T_n - \lambda)x_n\| \leq \|(T - \lambda)x\| + \|T_n x_n - Tx\| + |\lambda|\|x_n - x\| < \varepsilon.$$

Therefore, $\lambda \in \sigma_{\mathrm{app},\varepsilon}(T_n) \subset \sigma_\varepsilon(T_n)$ for all $n \geq n_\lambda$.

ii) By Lemma 3.2 ii), $\sigma_\varepsilon(T) = \sigma_{\mathrm{app},\varepsilon}(T) \cup \sigma_{\mathrm{app},\varepsilon}(T^*)^*$. Now the assertion follows from claim i) and the assumptions $T_n \stackrel{gsr}{\to} T$, $T_n^* \stackrel{gsr}{\to} T^*$. □

Now we confine the set of pseudospectral pollution.

Proposition 3.16. *Suppose that $T_n \stackrel{gsr}{\to} T$ and $T_n^* \stackrel{gsr}{\to} T^*$. Let $\lambda \in \varrho(T) \cap \Delta_b\big((T_n)_{n\in\mathbb{N}}\big)$ and let $\lambda_n \in \mathbb{C}$, $n \in \mathbb{N}$, satisfy $\lambda_n \to \lambda$, $n \to \infty$. Then*

$$M := \limsup_{n\to\infty} \|(T_n - \lambda_n)^{-1}\| = \limsup_{n\to\infty} \|(T_n - \lambda)^{-1}\| \geq \|(T - \lambda)^{-1}\|;$$

if the inequality is strict, then

$$\lambda \in \Lambda_{\mathrm{ess},\frac{1}{M}}\big((T_n)_{n\in\mathbb{N}}\big) \cap \Lambda_{\mathrm{ess},\frac{1}{M}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*.$$

Proof. First we prove that

$$M = \limsup_{n\to\infty} \|(T_n - \lambda_n)^{-1}\| = \limsup_{n\to\infty} \|(T_n - \lambda)^{-1}\|. \tag{3.8}$$

Since $\lambda \in \Delta_b\big((T_n)_{n\in\mathbb{N}}\big)$, we have $C := \sup_{n\in\mathbb{N}} \|(T_n - \lambda)^{-1}\| < \infty$. A Neumann series argument yields that, for all $n \in \mathbb{N}$ so large that $|\lambda_n - \lambda| < 1/C$,

$$\|(T_n - \lambda_n)^{-1}\| = \big\|(T_n - \lambda)^{-1}(I - (\lambda_n - \lambda)(T_n - \lambda)^{-1})^{-1}\big\| \leq \frac{C}{1 - |\lambda_n - \lambda|C}.$$

The first resolvent identity implies

$$\begin{aligned}\big|\|(T_n - \lambda)^{-1}\| - \|(T_n - \lambda_n)^{-1}\|\big| &\leq |\lambda - \lambda_n|\|(T_n - \lambda_n)^{-1}\|\|(T_n - \lambda)^{-1}\| \\ &\leq |\lambda - \lambda_n|\frac{C^2}{1 - |\lambda_n - \lambda|C}.\end{aligned}$$

The right hand side converges to 0 since $\lambda_n \to \lambda$. This proves (3.8).

The inequality

$$\limsup_{n\to\infty} \|(T_n - \lambda)^{-1}\| \geq \|(T - \lambda)^{-1}\|$$

follows from Theorem 3.3 ii).

Now assume that $M > \|(T - \lambda)^{-1}\|$. First note that $(T_n^* - \overline{\lambda})^{-1}(T_n - \lambda)^{-1}$ is selfadjoint and

$$\|(T_n-\lambda)^{-1}\|^2 = \sup_{\|y\|=1} \langle (T_n^*-\overline{\lambda})^{-1}(T_n-\lambda)^{-1}y, y\rangle = \max \sigma_{\mathrm{app}}\big((T_n^*-\overline{\lambda})^{-1}(T_n-\lambda)^{-1}\big).$$

Therefore,

$$M \in \sigma_{\mathrm{app}}\Big(\big((T_n^* - \overline{\lambda})^{-1}(T_n - \lambda)^{-1}\big)_{n\in\mathbb{N}}\Big).$$

By the assumptions $T_n \stackrel{gsr}{\to} T$, $T_n^* \stackrel{gsr}{\to} T^*$ and $\lambda \in \Delta_b\big((T_n)_{n\in\mathbb{N}}\big) \cap \varrho(T)$, we obtain, using [3, Proposition 2.16],

$$(T_n^* - \overline{\lambda})^{-1}(T_n - \lambda)^{-1}P_n \xrightarrow{s} (T^* - \overline{\lambda})^{-1}(T - \lambda)^{-1}P, \quad n \to \infty.$$

Moreover, [3, Lemma 3.2] yields $(T_n^* - \overline{\lambda})^{-1}(T_n - \lambda)^{-1} \stackrel{gsr}{\to} (T^* - \overline{\lambda})^{-1}(T - \lambda)^{-1}$. Now Proposition 2.15 ii) implies that

$$M^2 \in \sigma_p\big((T^* - \overline{\lambda})^{-1}(T - \lambda)^{-1}\big) \cup \sigma_{\mathrm{ess}}\Big(\big((T_n^* - \overline{\lambda})^{-1}(T_n - \lambda)^{-1}\big)_{n\in\mathbb{N}}\Big).$$

First case: If $M^2 \in \sigma_p\big((T^*-\overline{\lambda})^{-1}(T-\lambda)^{-1}\big)$, then there exists $y \in H$ with $\|y\| = 1$ such that

$$0 = \big\langle\big((T^*-\overline{\lambda})^{-1}(T-\lambda)^{-1} - M^2\big)y, y\big\rangle = \|(T-\lambda)^{-1}y\|^2 - M^2.$$

So we arrive at the contradiction $\|(T-\lambda)^{-1}\| \geq M$.

Second case: If $M^2 \in \sigma_{\rm ess}\big(\big((T_n^*-\overline{\lambda})^{-1}(T_n-\lambda)^{-1}\big)_{n\in\mathbb{N}}\big)$, then the mapping result in Theorem 2.5 implies that $1/M^2 \in \sigma_{\rm ess}\big(((T_n-\lambda)(T_n^*-\overline{\lambda}))_{n\in\mathbb{N}}\big)$. Hence there exist an infinite subset $I \subset \mathbb{N}$ and $x_n \in \mathcal{D}((T_n-\lambda)(T_n^*-\overline{\lambda})) \subset \mathcal{D}(T_n^*)$, $n \in I$, with $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and $\|((T_n-\lambda)(T_n^*-\overline{\lambda}) - 1/M^2)x_n\| \to 0$. So we arrive at

$$\|(T_n^*-\overline{\lambda})x_n\|^2 = \langle (T_n-\lambda)(T_n^*-\overline{\lambda})x_n, x_n\rangle \longrightarrow \frac{1}{M^2}, \quad n \in I, \quad n \to \infty.$$

This implies $\lambda \in \Lambda_{{\rm ess},1/M}\big((T_n^*)_{n\in\mathbb{N}}\big)^*$.

Since $\|(T-\lambda)^{-1}\| = \|(T^*-\overline{\lambda})^{-1}\|$ and $\|(T_n-\lambda)^{-1}\| = \|(T_n^*-\overline{\lambda})^{-1}\|$, we obtain analogously that $\lambda \in \Lambda_{{\rm ess},1/M}\big((T_n)_{n\in\mathbb{N}}\big)$. $\square$

Next we prove the ε-pseudospectral exactness result.

Proof of Theorem 3.6. i) Let $\lambda \in \overline{\sigma_\varepsilon(T)}$. Assume that the claim is false, i.e.

$$\alpha := \limsup_{n\to\infty} {\rm dist}(\lambda, \overline{\sigma_\varepsilon(T_n)}) > 0. \tag{3.9}$$

Choose $\widetilde{\lambda} \in \sigma_\varepsilon(T)$ with $|\lambda - \widetilde{\lambda}| < \alpha/2$. By Theorem 3.3 ii), there exists $n_{\widetilde{\lambda}} \in \mathbb{N}$ such that $\widetilde{\lambda} \in \sigma_\varepsilon(T_n) \subset \overline{\sigma_\varepsilon(T_n)}$, $n \geq n_{\widetilde{\lambda}}$, which is a contradiction to (3.9).

ii) Choose $\lambda \in \mathbb{C}\backslash\overline{\sigma_\varepsilon(T)}$ outside the set in (3.1) or (3.2), respectively. Assume that it is a point of ε-pseudospectral pollution, i.e. there exist an infinite subset $I \subset \mathbb{N}$ and $\lambda_n \in \overline{\sigma_\varepsilon(T_n)}$, $n \in I$, with $\lambda_n \to \lambda$. By the choice of λ and Proposition 2.16 i), we arrive at $\lambda \in \varrho(T) \cap \Delta_b\big((T_n)_{n\in\mathbb{N}}\big)$.

Since $\lambda_n \in \overline{\sigma_\varepsilon(T_n)}$, $n \in I$, we have

$$M := \limsup_{\substack{n\in I\\ n\to\infty}} \|(T_n-\lambda_n)^{-1}\| \geq \frac{1}{\varepsilon}.$$

Now, by Proposition 3.16 and using $\lambda \notin \Lambda_{{\rm ess},(0,\varepsilon]}$, we conclude that $\|(T-\lambda)^{-1}\| = 1/\varepsilon$. By Theorem 3.8 ii), the level set $\{\lambda \in \varrho(T) : \|(T-\lambda)^{-1}\| = 1/\varepsilon\}$ does not have an open subset. Hence we arrive at the contradiction $\lambda \in \overline{\sigma_\varepsilon(T)}$, which proves the claim.

iii) The proof is similar to the one of Theorem 2.3 iii). Assume that the claim is false. Then there exist $\alpha > 0$, an infinite subset $I \subset \mathbb{N}$ and $\lambda_n \in K$, $n \in I$, such that one of the following holds:

(1) $\lambda_n \in \overline{\sigma_\varepsilon(T_n)}$ and ${\rm dist}(\lambda_n, \overline{\sigma_\varepsilon(T)} \cap K) > \alpha$ for every $n \in I$;
(2) $\lambda_n \in \overline{\sigma_\varepsilon(T)}$ and ${\rm dist}(\lambda_n, \overline{\sigma_\varepsilon(T_n)} \cap K) > \alpha$ for every $n \in I$.

Note that, in both cases (1) and (2), the compactness of K implies that there exist $\lambda \in K$ and an infinite subset $J \subset I$ such that $(\lambda_n)_{n\in J}$ converges to λ.

First we consider case (1). Claim ii) and the assumptions on K imply that $\lambda \in \overline{\sigma_\varepsilon(T)} \cap K$ and hence $|\lambda_n - \lambda| \geq {\rm dist}(\lambda_n, \overline{\sigma_\varepsilon(T)} \cap K) > \alpha$, $n \in J$, a contradiction to $\lambda_n \to \lambda$.

Now assume that (2) holds. The assumption $\overline{\sigma_\varepsilon(T)} \cap K = \overline{\sigma_\varepsilon(T) \cap K}$ implies that $(\lambda_n)_{n\in J} \subset \overline{\sigma_\varepsilon(T) \cap K}$ and thus $\lambda \in \overline{\sigma_\varepsilon(T) \cap K}$. Choose $\widetilde{\lambda} \in \sigma_\varepsilon(T) \cap K$ with $|\lambda - \widetilde{\lambda}| < \alpha/2$. By Theorem 3.3 ii), there exists $n_{\widetilde{\lambda}} \in \mathbb{N}$ such that $\widetilde{\lambda} \in \sigma_\varepsilon(T_n) \cap K$ for every $n \geq n_{\widetilde{\lambda}}$. Therefore $|\lambda_n - \widetilde{\lambda}| \geq {\rm dist}(\lambda_n, \overline{\sigma_\varepsilon(T_n)} \cap K) > \alpha$ for every $n \in J$ with $n \geq n_{\widetilde{\lambda}}$. Since $|\lambda_n - \widetilde{\lambda}| \to |\lambda - \widetilde{\lambda}| < \alpha/2$, we arrive at a contradiction. This proves the claim. $\square$

Finally we prove the result about operators that have constant resolvent norm on an open set.

Proof of Theorem 3.8. i) Let $\lambda_0 \in U$. By proceeding as in the proof of Theorem 3.6 (with $T_n = T$, $\lambda_n = \lambda = \lambda_0$, $n \in \mathbb{N}$, and $M = 1/\varepsilon$), we obtain

$$\frac{1}{\varepsilon^2} \in \sigma_p((T^* - \overline{\lambda_0})^{-1}(T-\lambda_0)^{-1}) \cup \sigma_{\mathrm{ess}}((T^* - \overline{\lambda_0})^{-1}(T-\lambda_0)^{-1}),$$

and the second case $1/\varepsilon^2 \in \sigma_{\mathrm{ess}}((T^* - \overline{\lambda_0})^{-1}(T-\lambda_0)^{-1})$ implies $\lambda_0 \in \Lambda_{\mathrm{ess},\varepsilon}(T^*)^*$. If however $1/\varepsilon^2 \in \sigma_p((T^* - \overline{\lambda_0})^{-1}(T-\lambda_0)^{-1})$, then there exists $y \in H$ with $\|y\| = 1$ such that $\|(T-\lambda)^{-1}y\| = 1/\varepsilon$. Hence $x := (T-\lambda)^{-1}y \neq 0$ satisfies $\|(T-\lambda)x\|/\|x\| = \varepsilon$. Note that $\mu \mapsto \|(T-\mu)x\|$ is a non-constant subharmonic function on $\mathbb{C}$ and thus satisfies the maximum principle. Therefore, in every open neighbourhood of λ_0 there exist points μ such that $\|(T-\mu)x\|/\|x\| < \varepsilon$ and thus $\lambda_0 \in \overline{\sigma_\varepsilon(T)}$, a contradiction.

Since $\|(T-\lambda)^{-1}\| = \|(T^* - \overline{\lambda})^{-1}\| = 1/\varepsilon$ for every $\lambda \in U$, we analogously obtain $\lambda_0 \in \Lambda_{\mathrm{ess},\varepsilon}(T)$. So there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and $\|(T-\lambda_0)x_n\| \to \varepsilon$. Define

$$e_n := \frac{(T-\lambda_0)x_n}{\|(T-\lambda_0)x_n\|}, \quad n \in \mathbb{N}.$$

Then $\|e_n\| = 1$ and $e_n \xrightarrow{w} 0$ by Lemma 3.13 applied to $T_n = T$. In addition, $\|(T-\lambda_0)^{-1}e_n\| \to \|(T-\lambda_0)^{-1}\| = 1/\varepsilon$. Analogously as in the proof of [4, Theorem 3.2], using the old results [19, Lemmas 1.1,3.0] by Globevnik and Vidav, one may show that

$$(T-\lambda_0)^{-2}e_n \longrightarrow 0, \quad n \to \infty; \tag{3.10}$$

note that [4, Theorem 3.2] was proved for complex uniformly convex Banach spaces and is thus, in particular, valid for Hilbert spaces (see [18] for a discussion about complex uniform convexity).

Let $\lambda \in \mathbb{C}\backslash \bigcap_{K \text{ compact}} \sigma(T+K)$. Then there exists a compact operator $K \in L(H)$ such that $\lambda \in \varrho(T+K)$. The second resolvent identity applied twice yields

$$\begin{aligned}
&(T+K-\lambda)^{-1} - (T-\lambda_0)^{-1} \\
&= (T+K-\lambda)^{-1}(-K+\lambda-\lambda_0)(T-\lambda_0)^{-1} \\
&= -\big(I + (T+K-\lambda)^{-1}(-K+\lambda-\lambda_0)\big)(T-\lambda_0)^{-1}K(T-\lambda_0)^{-1} \\
&\quad + (\lambda-\lambda_0)\big(I + (T+K-\lambda)^{-1}(-K+\lambda-\lambda_0)\big)(T-\lambda_0)^{-2}.
\end{aligned}$$

Since $\widetilde{K} := -\big(I + (T+K-\lambda)^{-1}(-K+\lambda-\lambda_0)\big)(T-\lambda_0)^{-1}K(T-\lambda_0)^{-1}$ is compact and hence completely continuous, the weak convergence $e_n \xrightarrow{w} 0$ yields $\widetilde{K}e_n \to 0$. Using (3.10) in addition, we conclude $\big((T+K-\lambda)^{-1} - (T-\lambda_0)^{-1}\big)e_n \to 0$ and hence

$$\lim_{n\to\infty} \|(T+K-\lambda)^{-1}e_n\| = \lim_{n\to\infty} \|(T-\lambda_0)^{-1}e_n\| = \frac{1}{\varepsilon}.$$

Now define

$$w_n := \frac{(T+K-\lambda)^{-1}e_n}{\|(T+K-\lambda)^{-1}e_n\|}, \quad n \in \mathbb{N}.$$

Then $\|w_n\| = 1$, $w_n \xrightarrow{w} 0$ and $\|(T+K-\lambda)w_n\| = \|(T+K-\lambda)^{-1}e_n\|^{-1} \to \varepsilon$. Therefore, $\lambda \in \Lambda_{\mathrm{ess},\varepsilon}(T+K)$. Theorem 3.15 i) applied to $T_n = T$ and $B_n = K$ yields $\Lambda_{\mathrm{ess},\varepsilon}(T+K) = \Lambda_{\mathrm{ess},\varepsilon}(T)$. So arrive at $\lambda \in \Lambda_{\mathrm{ess},\varepsilon}(T)$.

In an analogous way as for (3.10), one may show that there exists a normalised sequence $(f_n)_{n\in\mathbb{N}} \subset H$ with $f_n \xrightarrow{w} 0$ and $(T^* - \overline{\lambda_0})^{-2}f_n \to 0$. So, by proceeding as above, we obtain $\lambda \in \Lambda_{\mathrm{ess},\varepsilon}(T^*)^*$.

ii) The claim follows from claim i) and Proposition 3.12 i). □

4. Applications and Examples

In this section we discuss applications to the Galerkin method for infinite matrices (Subsection 4.1) and to the domain truncation method for differential operators (Subsection 4.2).

4.1. **Galerkin approximation of block-diagonally dominant matrices.** In this subsection we consider an operator A in $l^2(\mathbb{K})$ (where $\mathbb{K} = \mathbb{N}$ or $\mathbb{K} = \mathbb{Z}$) whose matrix representation (identified with A) with respect to the standard orthonormal basis $\{e_j : j \in \mathbb{K}\}$ can be split as $A = T + S$. Here T is block-diagonal, i.e. there exist $m_k \in \mathbb{N}$ with

$$T = \operatorname{diag}(T_k : k \in \mathbb{K}), \quad T_k \in \mathbb{C}^{m_k \times m_k}.$$

We further assume that $\mathcal{D}(T) \subset \mathcal{D}(S)$, $\mathcal{D}(T^*) \subset \mathcal{D}(S^*)$ and that there exists $\lambda_0 \in \varrho(T)$ such that $S(T-\lambda_0)^{-1}$ is compact and

$$\|S(T-\lambda_0)^{-1}\| < 1, \quad \|S^*(T^*-\overline{\lambda_0})^{-1}\| < 1. \tag{4.1}$$

Define, for $n \in \mathbb{N}$,

$$j_n := \begin{cases} -\sum\limits_{k=-n}^{0} m_k, & \mathbb{K} = \mathbb{Z}, \\ 1, & \mathbb{K} = \mathbb{N}, \end{cases}, \qquad J_n := \sum_{k=1}^{n} m_k.$$

Let P_n be the orthogonal projection of $l^2(\mathbb{K})$ onto $H_n := \operatorname{span}\{e_j : j_n \le j \le J_n\}$. It is easy to see that $P_n \overset{s}{\to} I$.

Theorem 4.1. *Define $A_n := P_n A|_{H_n}$, $n \in \mathbb{N}$.*

i) *We have $A_n \overset{gsr}{\to} A$ and $A_n^* \overset{gsr}{\to} A^*$.*

ii) *The limiting essential spectra satisfy*

$$\begin{aligned} &\sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((A_n^*)_{n\in\mathbb{N}}\big)^* \\ &= \{\lambda \in \mathbb{C} : \exists\, I \subset \mathbb{N} \text{ with } \|(T_n-\lambda)^{-1}\| \to \infty,\, n \in I,\, n \to \infty\} \\ &= \sigma_{\mathrm{ess}}(A); \end{aligned}$$

hence no spectral pollution occurs for the approximation $(A_n)_{n\in\mathbb{N}}$ of A, and for every isolated $\lambda \in \sigma_{\mathrm{dis}}(A)$ there exists a sequence of $\lambda_n \in \sigma(A_n)$, $n \in \mathbb{N}$, with $\lambda_n \to \lambda$ as $n \to \infty$.

iii) *The limiting essential ε-near spectrum satisfies*

$$\begin{aligned} &\Lambda_{\mathrm{ess},\varepsilon}\big((A_n)_{n\in\mathbb{N}}\big) \\ &= \left\{\lambda \in \mathbb{C} : \exists\, I \subset \mathbb{N} \text{ with } \|(T_n-\lambda)^{-1}\| \to \frac{1}{\varepsilon},\, n \in I,\, n \to \infty\right\} \\ &= \Lambda_{\mathrm{ess},\varepsilon}(A); \end{aligned}$$

hence if A does not have constant resolvent norm ($= 1/\varepsilon$) on an open set, then $(A_n)_{n\in\mathbb{N}}$ is an ε-pseudospectrally exact approximation of A.

Proof. First note that the adjoint operators satisfy $A_n^* = T_n^* + S_n^*$, and since, by (4.1), S is T-bounded and S^* is T^*-bounded with relative bounds < 1, [22, Corollary 1] implies $A^* = T^* + S^*$. In addition, for any $n \in \mathbb{N}$,

$$\begin{aligned} (T_n - \lambda_0)^{-1} P_n &= P_n (T - \lambda_0)^{-1}, \\ (T_n^* - \overline{\lambda_0})^{-1} P_n &= P_n (T^* - \overline{\lambda_0})^{-1}, \\ S_n (T_n - \lambda_0)^{-1} P_n &= P_n S (T - \lambda_0)^{-1}, \\ S_n^* (T_n^* - \overline{\lambda_0})^{-1} P_n &= P_n S^* (T^* - \overline{\lambda_0})^{-1}, \\ (S_n (T_n - \lambda_0)^{-1})^* P_n|_{\mathcal{D}(S^*)} &= P_n (T^* - \overline{\lambda_0})^{-1} S^* = P_n (S(T - \lambda_0)^{-1})^*|_{\mathcal{D}(S^*)}. \end{aligned} \tag{4.2}$$

Now, using (4.2) everywhere, we check that the assumptions of Theorem 3.15 ii), iii) are satisfied.

(a) We readily conclude
$$\|S_n (T_n - \lambda_0)^{-1}\| \le \|S(T - \lambda_0)^{-1}\| < 1. \tag{4.3}$$
(b) The sequence of operators $S_n (T_n - \lambda_0)^{-1} = P_n S (T - \lambda_0)^{-1}|_{H_n}$, $n \in \mathbb{N}$, is discretely compact since $S(T - \lambda_0)^{-1}$ is compact and $P_n \stackrel{s}{\to} I$.
(c) The strong convergence $(T_n^* - \overline{\lambda_0})^{-1} P_n \stackrel{s}{\to} (T^* - \overline{\lambda_0})^{-1}$ follows immediately from $P_n \stackrel{s}{\to} I$. In addition, since $\mathcal{D}(S^*)$ is a dense subset, using (4.3) we obtain $(S_n (T_n - \lambda_0)^{-1})^* P_n \stackrel{s}{\to} (S(T - \lambda_0)^{-1})^*$.

Now Theorem 3.15 ii), iii) implies $A_n^* \stackrel{gsr}{\to} A^*$ and

$$\begin{aligned} &\sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big) = \sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big), \quad \Lambda_{\mathrm{ess},\varepsilon}\big((A_n)_{n\in\mathbb{N}}\big) = \Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big), \\ &\qquad \Lambda_{\mathrm{ess},\varepsilon}(A) = \Lambda_{\mathrm{ess},\varepsilon}(T). \end{aligned}$$

In addition, since $S(T - \lambda_0)^{-1}$ is assumed to be compact, [16, Theorem IX.2.1] yields $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(T)$.

In claim i) it is left to be shown that $A_n \stackrel{gsr}{\to} A$. To this end, we use that (4.2) and $P_n \stackrel{s}{\to} I$ imply

$$(T_n - \lambda_0)^{-1} P_n \stackrel{s}{\to} (T - \lambda_0)^{-1}, \quad S_n (T_n - \lambda_0)^{-1} P_n \stackrel{s}{\to} S(T - \lambda_0)^{-1}.$$

Now the claim follows from (4.3) and the perturbation result [3, Theorem 3.3].

Note that Theorem 2.12 ii) implies $\sigma_{\mathrm{ess}}\big((A_n^*)_{n\in\mathbb{N}}\big)^* = \sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^*$. The identities in claim ii) are obtained from

$$\begin{aligned} &\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \cup \sigma_{\mathrm{ess}}\big((T_n^*)_{n\in\mathbb{N}}\big)^* \\ &\quad = \{\lambda \in \mathbb{C} : \exists I \subset \mathbb{N} \text{ with } \|(T_n - \lambda)^{-1}\| \to \infty,\, n \in I,\, n \to \infty\} = \sigma_{\mathrm{ess}}(T). \end{aligned}$$

Now the local spectral exactness follows from Theorem 2.3.

The assertion in iii) follows from an analogous reasoning, using Theorem 3.6; note that if T does not have constant ($= 1/\varepsilon$) resolvent norm on an open set, then $\Lambda_{\mathrm{ess},\varepsilon}(T) \subset \partial\sigma_\varepsilon(T) \subset \overline{\sigma_\varepsilon(T)}$ and hence no ε-pseudospectral pollution occurs. □

Example 4.2. For points $b, d \in \mathbb{C}$ and sequences $(a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}}, (c_j)_{j\in\mathbb{N}}, (d_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ with

$$|a_j| \longrightarrow \infty, \quad b_j \longrightarrow b, \quad c_j \longrightarrow 0, \quad d_j \longrightarrow d, \quad j \to \infty,$$

define an unbounded operator A in $l^2(\mathbb{N})$ by

$$A := \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & d_1 & b_2 & & \\ & c_2 & a_2 & b_3 & \\ & & c_3 & d_2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{D}(A) := \Big\{ (x_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N}) : \sum_{j\in\mathbb{N}} |a_j x_{2j-1}|^2 < \infty \Big\}.$$

For $n \in \mathbb{N}$ let P_n be the orthogonal projection of $l^2(\mathbb{N})$ onto the first $2n$ basis vectors, and define $A_n := P_n A|_{\mathcal{R}(P_n)}$. Using Theorem 4.1, we show that

$$\sigma_{\mathrm{ess}}(A) = \{d\}, \quad \Lambda_{\mathrm{ess},\varepsilon}(A) = \{\lambda \in \mathbb{C} : |\lambda - d| = \varepsilon\},$$

that every $\lambda \in \sigma_{\mathrm{dis}}(A)$ is an accumulation point of $\sigma(A_n)$, $n \in \mathbb{N}$, that no spectral pollution occurs and that $(A_n)_{n\in\mathbb{N}}$ is ε-pseudospectrally exact.

To this end, define

$$T := \operatorname{diag}(T_k : k \in \mathbb{N}), \quad T_k := \begin{pmatrix} a_k & b_{2k-1} \\ 0 & d_k \end{pmatrix}, \quad \mathcal{D}(T) := \mathcal{D}(A).$$

Then it is easy to check that $S := A - T$ is T-compact and the estimates in (4.1) are satisfied for all $\lambda_0 \in \mathbb{C}$ that are sufficiently far from $\sigma(T)$. The essential spectrum $\sigma_{\mathrm{ess}}(T)$ consists of all accumulation points of $\sigma(T_k) = \{a_k, d_k\}$, $k \in \mathbb{N}$, i.e. $\sigma_{\mathrm{ess}}(T) = \{d\}$. To find $\Lambda_{\mathrm{ess},\varepsilon}(T)$, note that, in the limit $k \to \infty$,

$$\|(T_k - \lambda)^{-1}\| = \left\| \begin{pmatrix} (a_k - \lambda)^{-1} & -b_{2k-1}(a_k - \lambda)^{-1}(d_k - \lambda)^{-1} \\ 0 & (d_k - \lambda)^{-1} \end{pmatrix} \right\| \longrightarrow \frac{1}{|d - \lambda|}.$$

This proves $\Lambda_{\mathrm{ess},\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda - d| = \varepsilon\}$. Now the claims follow from Theorem 4.1 and since $\Lambda_{\mathrm{ess},\varepsilon}(T)$ does not contain an open subset.

The following example is influenced by Shargorodsky's example [31, Theorems 3.2, 3.3] of an operator with constant resolvent norm on an open set and whose matrix representation is block-diagonal. Here we perturb a block-diagonal operator with constant resolvent norm on an open set and arrive at an operator whose resolvent norm is also constant on an open set.

Example 4.3. Consider the *neutral delay differential expression* τ defined by

$$(\tau f)(t) := \mathrm{e}^{\mathrm{i}t}(f''(t) + f''(t + \pi)) + \mathrm{e}^{-\mathrm{i}t} f(t).$$

For an extensive treatment of neutral differential equations with delay, see the monograph [1] (in particular Chapter 3 for second order equations). Let A be the realisation of τ in $L^2(-\pi, \pi)$ with domain

$$\mathcal{D}(A) := \left\{ f \in L^2(-\pi, \pi) : \begin{array}{l} f, f' \in \mathrm{AC}_{\mathrm{loc}}(-\pi, \pi),\ \tau f \in L^2(-\pi, \pi), \\ f(-\pi) = f(\pi),\ f'(-\pi) = f'(\pi) \end{array} \right\},$$

where f is continued 2π-periodically. With respect to the orthonormal basis $\{e_k : k \in \mathbb{Z}\} \subset \mathcal{D}(A)$ with $e_k(t) := \mathrm{e}^{\mathrm{i}kt}/\sqrt{2\pi}$, the operator A has the matrix representation

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ & T_{-1} & S_{-1} & & \\ & & T_0 & S_0 & \\ & & & T_1 & \ddots \\ & & & & \ddots \end{pmatrix}, \quad T_j = \begin{pmatrix} 0 & 1 \\ a_j & 0 \end{pmatrix}, \quad S_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$a_j = 8j^2, \quad \mathcal{D}(A) = \left\{ ((u_j, v_j)^t)_{j\in\mathbb{Z}} : (u_j)_{j\in\mathbb{Z}}, (v_j)_{j\in\mathbb{Z}} \in l^2(\mathbb{Z}), \sum_{j\in\mathbb{Z}} |a_j u_j|^2 < \infty \right\}.$$

We split A to $T := \operatorname{diag}(T_j : j \in \mathbb{Z})$, $\mathcal{D}(T) := \mathcal{D}(A)$, and $S := A - T$. Note that S on $\mathcal{D}(S) = l^2(\mathbb{Z})$ is bounded and T-compact and S^* is T^*-compact, but T does not have compact resolvent. Next we prove the existence of $\lambda_0 \in \varrho(T)$ such that the estimates in (4.1) are satisfied. To this end, let $\lambda_0 \in \mathrm{i}\mathbb{R}\backslash\{0\}$ and estimate

$$\|S(T - \lambda_0)^{-1}\| = \sup_{j\in\mathbb{Z}} \|S_j(T_{j+1} - \lambda_0)^{-1}\| = \sup_{j\in\mathbb{Z}} \left\| \frac{1}{a_j - \lambda_0^2} \begin{pmatrix} 0 & 0 \\ \lambda_0 & 1 \end{pmatrix} \right\| \le \frac{1 + |\lambda_0|}{|\lambda_0|^2};$$

one may check that also $\|S^*(T^*-\overline{\lambda_0})^{-1}\| \leq (1+|\lambda_0|)/|\lambda_0|^2$. Hence (4.1) is satisfied if $|\lambda_0|$ is sufficiently large.

Let P_n denote the orthogonal projection onto

$$H_n := \operatorname{span}\{e_k : k = -(2n), \dots, 2n-1\},$$

and let A_n and T_n denote the respective Galerkin approximations,

$$A_n := P_n A|_{H_n}, \quad T_n := P_n T|_{H_n}, \quad n \in \mathbb{N}.$$

Note that $\det(A_n - \lambda) = \det(T_n - \lambda)$ for every $\lambda \in \mathbb{C}$, which implies $\sigma(A_n) = \sigma(T_n) = \big\{ \pm\sqrt{a_j} : j = -n, \dots, n \big\}$. Hence Theorem 4.1 ii) proves

$$\sigma(A) = \big\{ \pm\sqrt{a_j} : j \in \mathbb{Z} \big\} = \{\pm\sqrt{8}\, j : j \in \mathbb{N}_0\}.$$

Now we study the pseudospectra of A and T. In Figure 2 the eigenvalues (blue dots) and nested ε-pseudospectra (different shades of grey) of A_n are shown for $n = 2, 4, 6$ and $\varepsilon = 0.5, 0.6, \dots, 1.4, 1.5$. As n is increased, for $\varepsilon > 1$ the ε-pseudospectra grow and seem to fill the whole complex plane, whereas for $\varepsilon \leq 1$ they converge to $\sigma_\varepsilon(A) \neq \mathbb{C}$. We prove these observations more rigorously. In fact, we show that there exists an open subset of the complex plane where the resolvent norms of A and T are constant ($1/\varepsilon = 1$). So we cannot conclude ε-pseudospectral exactness using Theorem 4.1 iii). However, ε-pseudospectral inclusion follows from Theorem 3.6 i). In addition, the upper block-triangular form of A implies that if $x \in H_n$, then $P_n(A-\lambda)^{-1}x = (A_n - \lambda)^{-1}x_n$. This yields $\|(A_n-\lambda)^{-1}\| \leq \|(A-\lambda)^{-1}\|$ and so

$$\overline{\sigma_\varepsilon(A_n)} \subset \overline{\sigma_\varepsilon(A)}, \quad n \in \mathbb{N}.$$

Hence no ε-pseudospectral pollution occurs.

We calculate, for $\lambda = r\mathrm{e}^{\mathrm{i}\varphi}$ with $\operatorname{Re}(\lambda^2) = r^2\cos(2\varphi) < 0$,

$$\|(T_j-\lambda)^{-1}\|^2 = \frac{1}{|a_j-\lambda^2|^2}\left\|\begin{pmatrix}\lambda & 1\\ a_j & \lambda\end{pmatrix}\right\|^2 \leq \frac{(r+\max\{a_j,1\})^2}{r^4+2a_jr^2|\cos(2\varphi)|+a_j^2}. \tag{4.4}$$

Hence, as in [4, Example 3.7], the resolvent norm of T is constant on a non-empty open set,

$$\|(T-r\mathrm{e}^{\mathrm{i}\varphi})^{-1}\| = 1 \quad \text{if} \quad \cos(2\varphi) < 0, \quad r \geq \max\left\{\frac{1+\sqrt{5}}{2}, \frac{1}{|\cos(2\varphi)|}\right\}.$$

One may check that

$$\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(T^*)^* = \emptyset, \qquad \bigcap_{K \text{ compact}} \sigma(T+K) = \emptyset.$$

In addition, since $\|(T_j-\lambda)^{-1}\| \to 1$, $j \to \infty$, for any $\lambda \in \varrho(T)$, we have $\Lambda_{\mathrm{ess},\varepsilon}(T) = \Lambda_{\mathrm{ess},\varepsilon}(T^*)^* = \emptyset$, $\varepsilon \neq 1$. Therefore, using Theorems 3.8 i) and 4.1,

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A^*)^* = \emptyset, \quad \Lambda_{\mathrm{ess},\varepsilon}(A) = \Lambda_{\mathrm{ess},\varepsilon}(A^*)^* = \begin{cases}\mathbb{C}, & \varepsilon = 1,\\ \emptyset, & \varepsilon \neq 1.\end{cases}$$

This implies, in particular, that $\|(A-\lambda)^{-1}\| \geq 1$ for all $\lambda \in \varrho(A)$. Now we prove that the resolvent norm is constant ($= 1$) on an open set. To this end, let φ be so that $\cos(2\varphi) < -\frac{1}{4}$. We show that there exists $r_\varphi > 0$ such that

$$\|(A-r\mathrm{e}^{\mathrm{i}\varphi})^{-1}\| = 1, \quad r \geq r_\varphi. \tag{4.5}$$

Define

$$\delta_{-1} := \frac{1}{3}, \quad \delta_j := \frac{1}{a_{j+1}|\cos(2\varphi)|}, \quad j \in \mathbb{Z}\backslash\{-1\}.$$

Then $\delta_j \to 0$ as $|j| \to \infty$ and

$$\delta_{j-1} a_j = \frac{1}{|\cos(2\varphi)|}, \quad j \in \mathbb{Z}\backslash\{0\}, \quad \delta := \sup_{j\in\mathbb{Z}} \delta_j = \max\left\{\frac{1}{3}, \frac{1}{a_1|\cos(2\varphi)|}\right\} < \frac{1}{2}.$$

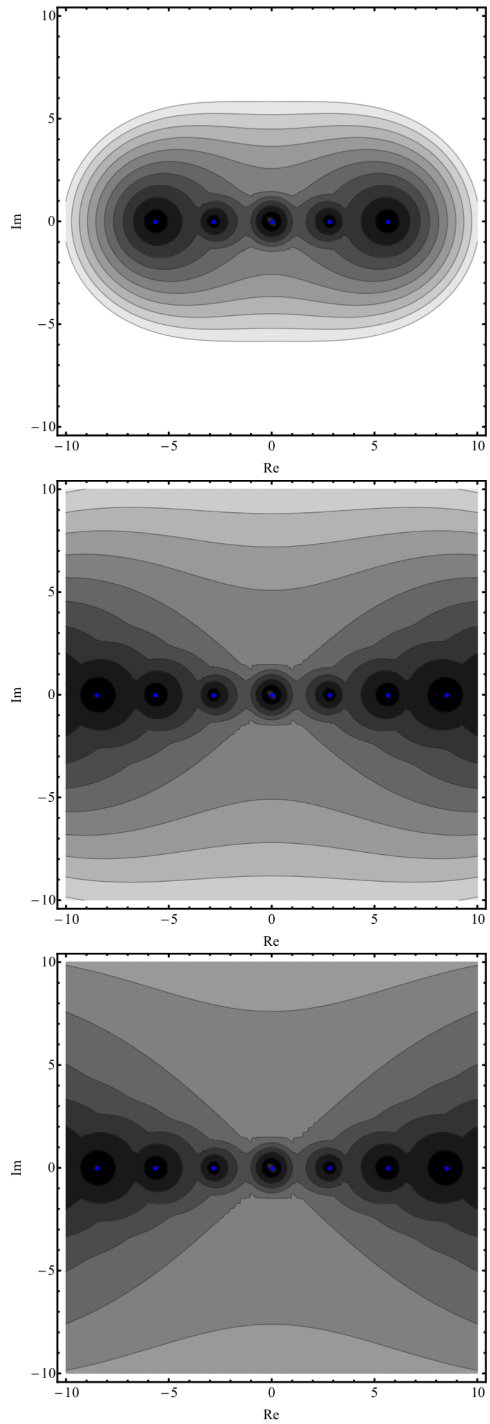


FIGURE 2. Eigenvalues (blue dots) and ε-pseudospectra of the truncated $4n \times 4n$ matrices A_n for $n = 2$ (top), $n = 4$ (middle), $n = 6$ (bottom) and $\varepsilon = 1.5, 1.4, \ldots, 0.6, 0.5$.

Define functions $f_j : [0,\infty) \to \mathbb{R}$, $j \in \mathbb{Z}$, by

$$f_0(r) := r^4(1-\delta) - \frac{r^2+1}{\delta_{-1}} - 2r$$

and, for $j \neq 0$,

$$\begin{aligned} f_j(r) := & r^4(1-\delta) + 1 \\ & + a_j r^2 \left(1 - 2\delta)|\cos(2\varphi)| - \frac{2}{r} - \sup_{j\in\mathbb{Z}\setminus\{-1\}} \frac{a_j}{a_{j+1}} \frac{1}{|\cos(2\varphi)|} - |\cos(2\varphi)|\right). \end{aligned}$$

One may verify that there exists $r_\varphi > 0$ such that $f_j(r) > 0$ for all $j \in \mathbb{Z}$ and $r \geq r_\varphi$. We calculate, for $\lambda = r\mathrm{e}^{\mathrm{i}\varphi}$ with $r \geq r_\varphi$,

$$\|S_{j-1}(T_j-\lambda)^{-1}\|^2 = \frac{|\lambda|^2+1}{|a_j-\lambda^2|^2} = \frac{r^2+1}{r^4+2a_jr^2|\cos(2\varphi)|+a_j^2}, \quad j \in \mathbb{Z}.$$

We abbreviate

$$g_j(r) := r^4 + 2a_j r^2|\cos(2\varphi)| + a_j^2 > 0, \quad j \in \mathbb{Z}.$$

Then, with (4.4), we estimate for $j \neq 0$,

$$\begin{aligned} & 1 - \delta_j - \left(\frac{1}{\delta_{j-1}} - 1\right)\|S_{j-1}(T_j-\lambda)^{-1}\|^2 - \|(T_j-\lambda)^{-1}\|^2 \\ & = \frac{r^4(1-\delta_j) + 1 + a_j\left(r^2\left(2(1-\delta_j)|\cos(2\varphi)| - \frac{1}{\delta_{j-1}a_j} - \frac{2}{r}\right) - \delta_j a_j - \frac{1}{\delta_{j-1}a_j}\right)}{g_j(r)} \\ & \geq \frac{f_j(r)}{g_j(r)} > 0, \end{aligned}$$

and analogously for $j = 0$. So we arrive at

$$1 - \delta_j - \left(\frac{1}{\delta_{j-1}} - 1\right)\|S_{j-1}(T_j-\lambda)^{-1}\|^2 > \|(T_j-\lambda)^{-1}\|^2, \quad j \in \mathbb{Z}.$$

Let $x = (x_j)_{j\in\mathbb{Z}} \in \mathcal{D}(A) = \mathcal{D}(T)$ with $x_j = (u_j, v_j)^t \in \mathbb{C}^2$. Then

$$\begin{aligned} & \|(A-\lambda)x\|^2 \\ & = \|(T-\lambda)x + Sx\|^2 = \sum_{j\in\mathbb{Z}} \|(T_j-\lambda)x_j + S_jx_{j+1}\|^2 \\ & \geq \sum_{j\in\mathbb{Z}} (1-\delta_j)\|(T_j-\lambda)x_j\|^2 - \left(\frac{1}{\delta_j} - 1\right)\|S_jx_{j+1}\|^2 \\ & \geq \sum_{j\in\mathbb{Z}} (1-\delta_j)\|(T_j-\lambda)x_j\|^2 - \left(\frac{1}{\delta_j} - 1\right)\|S_j(T_{j+1}-\lambda)^{-1}\|^2\|(T_{j+1}-\lambda)x_{j+1}\|^2 \\ & = \sum_{j\in\mathbb{Z}} \left(1 - \delta_j - \left(\frac{1}{\delta_{j-1}} - 1\right)\|S_{j-1}(T_j-\lambda)^{-1}\|^2\right)\|(T_j-\lambda)x_j\|^2 \\ & \geq \sum_{j\in\mathbb{Z}} \|(T_j-\lambda)^{-1}\|^2\|(T_j-\lambda)x_j\|^2 \geq \|x\|^2, \end{aligned}$$

which implies $\|(A-\lambda)^{-1}\| \leq 1$ and hence (4.5).

4.2. **Domain truncation of PDEs on $\mathbb{R}^d$.** In this application we study the sum of two partial differential operators in $L^2(\mathbb{R}^d)$, the first one of order $k \in \mathbb{N}$ and the second one is of lower order and relatively compact. To this end, we use a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)^t \in \mathbb{R}^d$ with $|\alpha| := \alpha_1 + \dots + \alpha_d$ and

$$D^\alpha := \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}x_1^{\alpha_1} \cdots \mathrm{d}x_d^{\alpha_d}}, \quad \zeta^\alpha := \zeta_1^{\alpha_1} \cdots \zeta_d^{\alpha_d}, \quad \zeta = (\zeta_1, \dots, \zeta_d)^t \in \mathbb{R}^d.$$

The differential expressions are of the form

$$\tau := \tau_1 + \tau_2, \quad \tau_1 := \sum_{|\alpha| \le k} \frac{1}{\mathrm{i}^{|\alpha|}} c_\alpha D^\alpha, \quad \tau_2 := \sum_{|\alpha| \le k-1} \frac{1}{\mathrm{i}^{|\alpha|}} b_\alpha D^\alpha,$$

where $c_\alpha \in \mathbb{C}$. In order to reduce the technical difficulties, we assume that the functions $b_\alpha : \mathbb{R} \to \mathbb{C}$ are sufficiently smooth,

$$b_\alpha \in W^{|\alpha|,\infty}(\mathbb{R}^d), \quad |\alpha| \le k-1.$$

In addition, suppose that

$$\lim_{|x| \to \infty} D^\beta b_\alpha(x) = 0, \quad |\beta| \le |\alpha| \le k-1. \tag{4.6}$$

Define the *symbol* $p : \mathbb{R}^d \to \mathbb{C}$ and *principal symbol* $p_k : \mathbb{R}^d \to \mathbb{C}$ by

$$p(\zeta) := p_k(\zeta) + \sum_{|\alpha| \le k-1} c_\alpha \zeta^\alpha, \quad p_k(\zeta) := \sum_{|\alpha| = k} c_\alpha \zeta^\alpha.$$

We assume that p is elliptic, i.e.

$$p_k(\zeta) \ne 0, \quad \zeta \in \mathbb{R}^d \backslash \{0\}.$$

For $n \in \mathbb{N}$ let P_n be the orthogonal projection of $L^2(\mathbb{R}^d)$ onto $L^2\big((-n,n)^d\big)$, given by multiplication with the characteristic function $\chi_{(-n,n)^d}$. It is easy to see that $P_n \stackrel{s}{\to} I$.

Theorem 4.4. *Let A and A_n, $n \in \mathbb{N}$, be realisations of τ in $L^2(\mathbb{R})$ and $L^2\big((-n,n)^d\big)$, $n \in \mathbb{N}$, respectively, with domains*

$$\mathcal{D}(A) := W^{k,2}(\mathbb{R}^d),$$
$$\mathcal{D}(A_n) := \big\{ f \in W^{k,2}\big((-n,n)^d\big) : D^\alpha f|_{\{x_j=-n\}} = D^\alpha f|_{\{x_j=n\}},\ j \le d,\ |\alpha| \le k-1 \big\}.$$

i) *We have $A_n \stackrel{gsr}{\to} A$ and $A_n^* \stackrel{gsr}{\to} A^*$.*

ii) *The limiting essential spectra satisfy*

$$\sigma_{\mathrm{ess}}\big((A_n)_{n\in\mathbb{N}}\big) = \sigma_{\mathrm{ess}}\big((A_n^*)_{n\in\mathbb{N}}\big)^* = \sigma_{\mathrm{ess}}(A) = \{p(\zeta) : \zeta \in \mathbb{R}^d\}; \tag{4.7}$$

hence no spectral pollution occurs for the approximation $(A_n)_{n\in\mathbb{N}}$ of A, and every isolated $\lambda \in \sigma_{\mathrm{dis}}(A)$ is the limit of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \sigma(A_n)$, $n \in \mathbb{N}$.

iii) *The limiting essential ε-near spectra satisfy*

$$\begin{aligned} \Lambda_{\mathrm{ess},\varepsilon}\big((A_n)_{n\in\mathbb{N}}\big) &= \Lambda_{\mathrm{ess},\varepsilon}\big((A_n^*)_{n\in\mathbb{N}}\big)^* = \Lambda_{\mathrm{ess},\varepsilon}(A) \\ &= \{p(\zeta) + z : \zeta \in \mathbb{R}^d,\ |z| = \varepsilon\} \subset \overline{\sigma_\varepsilon(A)}, \end{aligned} \tag{4.8}$$

and so $(A_n)_{n\in\mathbb{N}}$ is an ε-pseudospectrally exact approximation of A.

Proof. Let T, S and T_n, S_n, $n \in \mathbb{N}$, be the realisations of τ_1, τ_2 in $L^2(\mathbb{R})$ and $L^2\big((-n,n)^d\big)$, $n \in \mathbb{N}$, respectively, with domains

$$\mathcal{D}(T) = \mathcal{D}(S) := \mathcal{D}(A), \quad \mathcal{D}(T_n) = \mathcal{D}(S_n) := \mathcal{D}(A_n), \quad n \in \mathbb{N}.$$

The operators T and T_n, $n \in \mathbb{N}$, are normal; the symbol of the adjoint operators T^*, T_n^*, $n \in \mathbb{N}$, is simply the complex conjugate symbol $\overline{p}$. For $f \in L^2(\mathbb{R}^d)$ denote its Fourier transform by $\widehat{f}$ and for $n \in \mathbb{N}$ and $f_n \in L^2((-n,n)^d)$ denote by $\widehat{f_n} = (\widehat{f_n}(\zeta))_{\zeta\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$ the complex Fourier coefficients, i.e.

$$\widehat{f}(\zeta) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}\zeta\cdot x} \mathrm{d}x, \quad \zeta \in \mathbb{R}^d,$$
$$\widehat{f_n}(\zeta) := \frac{1}{(2n)^{\frac{d}{2}}} \int_{(-n,n)^d} f(x) \mathrm{e}^{-\mathrm{i}\frac{\pi}{n}\zeta\cdot x} \mathrm{d}x, \quad \zeta \in \mathbb{Z}^d.$$

Parseval's identity yields that, if $f \in \mathcal{D}(T)$, $f_n \in \mathcal{D}(T_n)$,

$$\|(T-\lambda)f\| = \|(p-\lambda)\widehat{f}\|, \quad \|(T_n-\lambda)f_n\| = \left\| \left(p\left(\cdot\frac{\pi}{n}\right) - \lambda\right) \widehat{f_n} \right\|_{l^2(\mathbb{Z}^d)};$$

moreover, if $\lambda \notin \{p(\zeta) : \zeta \in \mathbb{R}^d\}$, then

$$\|(T-\lambda)^{-1}f\| = \|(p-\lambda)^{-1}\widehat{f}\|, \quad \|(T_n-\lambda)^{-1}f_n\| = \left\| \left(p\left(\cdot\frac{\pi}{n}\right) - \lambda\right)^{-1} \widehat{f_n} \right\|_{l^2(\mathbb{Z}^d)}.$$

We readily conclude

$$\sigma_{\mathrm{ess}}(T) = \sigma(T) = \{p(\zeta) : \zeta \in \mathbb{R}^d\},$$
$$\sigma_\varepsilon(T) = \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(T)) < \varepsilon\}, \quad \Lambda_{\mathrm{ess},\varepsilon}(T) = \{p(\zeta) + z : \zeta \in \mathbb{R}^d, |z| = \varepsilon\},$$
$$\sigma(T_n) = \left\{p\left(\zeta\frac{\pi}{n}\right) : \zeta \in \mathbb{Z}^d\right\}, \quad \sigma_\varepsilon(T_n) = \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma(T_n)) < \varepsilon\}, \quad n \in \mathbb{N},$$

and

$$\sigma_{\mathrm{ess}}\big((T_n)_{n\in\mathbb{N}}\big) \subset \{p(\zeta) : \zeta \in \mathbb{R}^d\} = \sigma_{\mathrm{ess}}(T),$$
$$\Lambda_{\mathrm{ess},\varepsilon}\big((T_n)_{n\in\mathbb{N}}\big) \subset \{p(\zeta) + z : \zeta \in \mathbb{R}^d, |z| = \varepsilon\} = \Lambda_{\mathrm{ess},\varepsilon}(T),$$

and the latter are equalities by Propositions 2.7 and 3.12. The same identities hold for the adjoint operators. This proves (4.7) and (4.8).

For any $\Omega \subset \mathbb{R}^d$ we have

$$\begin{aligned} \|\chi_\Omega S(T-\lambda)^{-1}f\| &\leq \sum_{|\alpha|\leq k-1} \|b_\alpha\|_{L^\infty(\Omega)} \sup_{\zeta\in\mathbb{R}^d} \left|\zeta^\alpha \Big(p\Big(\zeta\frac{\pi}{n}\Big) - \lambda\Big)^{-1}\right| \|f\|, \\ \|\chi_\Omega S_n(T_n-\lambda)^{-1}f_n\| &\leq \sum_{|\alpha|\leq k-1} \|b_\alpha\|_{L^\infty(\Omega)} \sup_{\zeta\in\mathbb{R}^d} \left|\zeta^\alpha \Big(p\Big(\zeta\frac{\pi}{n}\Big) - \lambda\Big)^{-1}\right| \|f_n\|. \end{aligned} \tag{4.9}$$

By setting $\Omega = \mathbb{R}^d$, we see that $\|S(T-\lambda)^{-1}\|$, $\|S_n(T_n-\lambda)^{-1}\|$, $n \in \mathbb{N}$, are uniformly bounded, and the uniform bound can be arbitrarily small by choosing λ far away from $\{p(\zeta) : \zeta \in \mathbb{R}^d\}$. The same argument also holds for the adjoint operators. Let λ_0 be so that

$$\begin{aligned} \|S(T-\lambda_0)^{-1}\| < 1, &\quad \sup_{n\in\mathbb{N}} \|S_n(T_n-\lambda_0)^{-1}\| < 1, \\ \|S^*(T^*-\overline{\lambda_0})^{-1}\| < 1, &\quad \sup_{n\in\mathbb{N}} \|S_n^*(T_n^*-\overline{\lambda_0})^{-1}\| < 1. \end{aligned} \tag{4.10}$$

Hence, in particular, S, S_n, S^*, S_n^* are respectively T-, T_n-, T^*-, T_n^*-bounded with relative bounds < 1 and so, by [22, Corollary 1], $A^* = T^* + S^*$ and $A_n^* = T_n^* + S_n^*$.

By the assumptions (4.6) and [16, Theorem IX.8.2], the operator S is T-compact and S^* is T^*-compact. Hence [16, Theorem IX.2.1] implies

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(T), \quad \sigma_{\mathrm{ess}}(A^*)^* = \sigma_{\mathrm{ess}}(T^*)^*.$$

Now we show that the assumptions (a)–(c) of Theorem 3.15 ii), iii) are satisfied for both $A = T + S$, $A_n = T_n + S_n$ and $A^* = T^* + S^*$, $A_n^* = T_n^* + S_n^*$; then the claims i)–iii) follow from the above arguments and together with Theorems 2.3, 3.6. We prove (c) before (b) as the proof of the latter relies on the former.

(a) The estimates are satisfied by the choice of λ_0, see (4.10).

(c) Let $f \in C_0^\infty(\mathbb{R}^d)$, and let $n_f \in \mathbb{N}$ be so large that $\operatorname{supp} f \subset (-n_f, n_f)^d$. Then, for $n \geq n_f$,

$$\begin{aligned}
(T_n - \lambda_0)^{-1} P_n f &= P_n (T - \lambda_0)^{-1} f, \\
(T_n^* - \overline{\lambda_0})^{-1} P_n f &= P_n (T^* - \overline{\lambda_0})^{-1} f, \\
(S_n (T_n - \lambda_0)^{-1})^* P_n f &= (T_n^* - \overline{\lambda_0})^{-1} S_n^* P_n f = P_n (T^* - \overline{\lambda_0})^{-1} S^* f \\
&= P_n (S (T - \lambda_0)^{-1})^* f, \\
(S_n^* (T_n^* - \overline{\lambda_0})^{-1})^* P_n f &= (T_n - \lambda_0)^{-1} S_n P_n f = P_n (T - \lambda_0)^{-1} S f \\
&= P_n (S^* (T^* - \overline{\lambda_0})^{-1})^* f.
\end{aligned}$$

Now the claimed strong convergences follow using (4.10) and the density of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$.

(b) We prove that $\big(S_n(T_n - \lambda_0)^{-1}\big)_{n\in\mathbb{N}}$ is a discretely compact sequence; for $\big(S_n^*(T_n^* - \overline{\lambda_0})^{-1}\big)_{n\in\mathbb{N}}$ the argument is analogous. To this end, let $I \subset \mathbb{N}$ be an infinite subset and let $f_n \in L^2((-n,n)^d)$, $n \in I$, be a bounded sequence. Then there exists an infinite subset $I_1 \subset I$ such that $(f_n)_{n\in I_1}$ is weakly convergent in $L^2(\mathbb{R}^d)$; denote the weak limit by f. We show that $\|S_n(T_n - \lambda_0)^{-1} f_n - S(T - \lambda_0)^{-1} f\| \to 0$ as $n \in I_1$, $n \to \infty$. Assume that the claim is false, i.e. there exist an infinite subset $I_2 \subset I_1$ and $\delta > 0$ so that

$$\|S_n(T_n - \lambda_0)^{-1} f_n - S(T - \lambda_0)^{-1} f\|^2 \geq \delta, \quad n \in I_2. \tag{4.11}$$

Note that (c) and Lemma 2.11 i) imply that $(S_n(T_n - \lambda)^{-1} f_n)_{n\in I_2}$ converges weakly to $S(T - \lambda_0)^{-1} f$. The assumption (4.6) yields

$$\lim_{n\to\infty} \|b_\alpha\|_{L^\infty(\mathbb{R}^d \setminus (-n,n)^d)} = 0, \quad |\alpha| \leq k - 1.$$

Hence, by (4.9), there exists $n_0 \in \mathbb{N}$ so large that

$$\|\chi_{\mathbb{R}^d\setminus(-n_0,n_0)^d} S_n (T_n - \lambda)^{-1} f_n - \chi_{\mathbb{R}^d\setminus(-n_0,n_0)^d} S(T - \lambda)^{-1} f\|^2 < \frac{\delta}{2}$$

for all $n \in I_2$ with $n \geq n_0$; denote by I_3 the set of all such n. An estimate similar to (4.9) reveals that the $W^{k,2}((-n_0,n_0)^d)$ norms

$$\|\chi_{(-n_0,n_0)^d} (T_n - \lambda_0)^{-1} f_n - \chi_{(-n_0,n_0)^d} (T - \lambda_0)^{-1} f\|_{W^{k,2}((-n_0,n_0)^d)}, \quad n \in I_3,$$

are uniformly bounded, and so the Sobolev embedding theorem yields

$$\|\chi_{(-n_0,n_0)^d} S_n (T_n - \lambda_0)^{-1} f_n - \chi_{(-n_0,n_0)^d} S(T - \lambda_0)^{-1} f\|_{L^2((-n_0,n_0)^d)} \longrightarrow 0.$$

Altogether we arrive at a contradiction to (4.11), which proves the claim. □

It is convenient to represent A_n with respect to the Fourier basis and, in a further approximation step, to truncate the infinite matrix to finite sections. We prove that these two approximation processes can be performed in one. For $n \in \mathbb{N}$ let

$$e_\zeta^{(n)}(x) := \frac{1}{(2n)^{\frac{d}{2}}} \, \mathrm{e}^{\mathrm{i}\frac{\pi}{n}\zeta\cdot x}, \quad x \in (-n,n)^d, \quad \zeta \in \mathbb{Z}^d,$$

denote the Fourier basis of $L^2((-n,n)^d)$. Note that the basis functions belong to $\mathcal{D}(A_n)$. Let Q_n denote the orthogonal projection of $L^2((-n,n)^d)$ onto

$$\operatorname{span}\big\{e_\zeta^{(n)} : \zeta \in \mathbb{Z}^d, \, \|\zeta\|_\infty \leq n\big\}. \tag{4.12}$$

One may check that $Q_n \xrightarrow{s} I$ and hence $Q_n P_n \xrightarrow{s} I$.

Theorem 4.5. *The claims* i)–iii) *of Theorem* 4.4 *continue to hold if A_n is replaced by $A_{n;n} := Q_n A_n|_{\mathcal{R}(Q_n)}$.*

Proof. Define $T_{n;n} := Q_n T_n|_{\mathcal{R}(Q_n)}$, $n \in \mathbb{N}$. Note that $Q_n T_n = T_{n;n} Q_n$, $n \in \mathbb{N}$. Analogously as in the proof of Theorem 4.4, we obtain

$$\sigma_{\text{ess}}\big((T_{n;n})_{n\in\mathbb{N}}\big) = \sigma_{\text{ess}}(T), \quad \Lambda_{\text{ess},\varepsilon}\big((T_{n;n})_{n\in\mathbb{N}}\big) = \Lambda_{\text{ess},\varepsilon}(T),$$

and the respective equalities for the adjoint operators. It is easy to see that $S_{n;n} := Q_n S_n|_{\mathcal{R}(Q_n)}$, $n \in \mathbb{N}$, satisfy

$$S_{n;n}(T_{n;n} - \lambda_0)^{-1} = Q_n S_n (T_n - \lambda)^{-1}|_{\mathcal{R}(Q_n)}, \quad n \in \mathbb{N},$$

and hence the discrete compactness of $\big(S_{n;n}(T_{n;n}-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}$ follows from the one of $\big(S_n(T_n-\lambda_0)^{-1}\big)_{n\in\mathbb{N}}$ and from $Q_n \xrightarrow{s} I$. By an analogous reasoning, the sequence $\big(S_{n;n}^*(T_{n;n}^* - \overline{\lambda_0}\big)^{-1})_{n\in\mathbb{N}}$ is discretely compact. The rest of the proof follows the one of Theorem 4.4. ☐

Example 4.6. Let $d = 1$ and consider the constant-coefficient differential operator

$$T := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2\,\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(T) := W^{2,2}(\mathbb{R}).$$

The above assumptions are satisfied if we perturb T by $S = b$ with a potential $b \in L^\infty(\mathbb{R})$ such that $|b(x)| \to 0$ as $|x| \to \infty$. For

$$b(x) := 20\sin(x)\mathrm{e}^{-x^2}, \quad x \in \mathbb{R},$$

the numerically found eigenvalues and pseudospectra of the operator $T+S$ truncated to the $(2n-1)$-dimensional subspace in (4.12) are shown in Figure 3. The

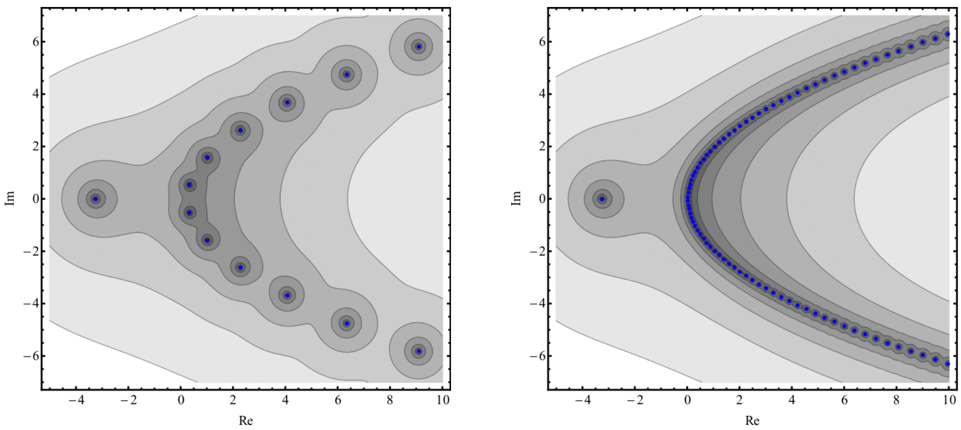


Figure 3. Eigenvalues (blue dots) and ε-pseudospectra for $\varepsilon = 2^3, 2^2, \ldots, 2^{-3}$ in interval $[-5,10]+[-7,7]\,\mathrm{i}$ of approximation $A_{n;n}$ for $n = 10$ (left) and $n = 100$ (right).

approximation is spectrally and ε-pseudospectrally exact; the only discrete eigenvalue in the box $[-5,10]+[-7,7]\,\mathrm{i}$ is $\lambda \approx -3.25$.

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References

[1] Baĭnov, D. D., and Mishev, D. P. *Oscillation theory for neutral differential equations with delay.* Adam Hilger, Ltd., Bristol, 1991.

[2] Bögli, S. Spectral approximation for linear operators and applications. Ph.D. thesis. 222 pages, 2014.

[3] Bögli, S. Convergence of sequences of linear operators and their spectra. arXiv:1604.07732, 2016.

[4] Bögli, S., and Siegl, P. Remarks on the convergence of pseudospectra. *Integral Equations Operator Theory 80* (2014), 303–321.

[5] Böttcher, A. Pseudospectra and singular values of large convolution operators. *J. Integral Equations Appl. 6*, 3 (1994), 267–301.

[6] Böttcher, A., and Silbermann, B. *Introduction to large truncated Toeplitz matrices.* Universitext. Springer-Verlag, New York, 1999.

[7] Böttcher, A., and Wolf, H. Spectral approximation for Segal-Bargmann space Toeplitz operators. In *Linear operators (Warsaw, 1994)*, vol. 38 of *Banach Center Publ.* Polish Acad. Sci., Warsaw, 1997, pp. 25–48.

[8] Boulton, L., Boussaïd, N., and Lewin, M. Generalised Weyl theorems and spectral pollution in the Galerkin method. *J. Spectr. Theory 2*, 4 (2012), 329–354.

[9] Brown, B. M., and Marletta, M. Spectral inclusion and spectral exactness for singular non-self-adjoint Sturm-Liouville problems. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 457* (2001), 117–139.

[10] Brown, B. M., and Marletta, M. Spectral inclusion and spectral exactness for singular non-self-adjoint Hamiltonian systems. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 459*, 2036 (2003), 1987–2009.

[11] Brown, B. M., and Marletta, M. Spectral inclusion and spectral exactness for PDEs on exterior domains. *IMA J. Numer. Anal. 24* (2004), 21–43.

[12] Chaitin-Chatelin, F., and Harrabi, A. About definitions of pseudospectra of closed operators in Banach spaces. Technical Report TR/PA/98/08, CERFACS, Toulouse, France, 1998.

[13] Chandler-Wilde, S. N., and Lindner, M. Limit operators, collective compactness, and the spectral theory of infinite matrices. *Mem. Amer. Math. Soc. 210*, 989 (2011), viii+111.

[14] Chatelin, F. *Spectral approximation of linear operators.* Academic Press Inc., New York, 1983.

[15] Davies, E. B. Pseudospectra of differential operators. *Journal of Operator Theory 43* (2000), 243–262.

[16] Edmunds, D. E., and Evans, W. D. *Spectral theory and differential operators.* Oxford University Press, New York, 1987.

[17] Fillmore, P. A., Stampfli, J. G., and Williams, J. P. On the essential numerical range, the essential spectrum, and a problem of Halmos. *Acta Sci. Math. (Szeged) 33* (1972), 179–192.

[18] Globevnik, J. On complex strict and uniform convexity. *Proceedings of the American Mathematical Society 47* (1975), 175–178.

[19] Globevnik, J., and Vidav, I. On operator-valued analytic functions with constant norm. *Journal of Functional Analysis 15* (1974), 394–403.

[20] Hansen, A. C. On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators. *J. Amer. Math. Soc. 24* (2011), 81–124.

[21] Harrabi, A. Pseudospectre d'une suite d'opérateurs bornés. *RAIRO Modélisation Mathématique et Analyse Numérique 32* (1998), 671–680.

[22] Hess, P., and Kato, T. Perturbation of closed operators and their adjoints. *Comment. Math. Helv. 45* (1970), 524–529.

[23] Kato, T. *Perturbation theory for linear operators.* Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[24] Levitin, M., and Shargorodsky, E. Spectral pollution and second-order relative spectra for self-adjoint operators. *IMA J. Numer. Anal. 24*, 3 (2004), 393–416.

[25] Lewin, M., and Séré, É. Spectral pollution and how to avoid it (with applications to Dirac and periodic Schrödinger operators). *Proc. Lond. Math. Soc. (3) 100*, 3 (2010), 864–900.

[26] Marletta, M. Neumann-Dirichlet maps and analysis of spectral pollution for non-self-adjoint elliptic PDEs with real essential spectrum. *IMA J. Numer. Anal. 30*, 4 (2010), 917–939.

[27] Osborn, J. E. Spectral approximation for compact operators. *Math. Comput. 29* (1975), 712–725.

[28] Reddy, S. C. Pseudospectra of Wiener-Hopf integral operators and constant-coefficient differential operators. *Journal of Integral Equations and Applications 5* (1993), 369–403.

[29] Reed, M., and Simon, B. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York-London, 1972.
[30] Roch, S., and Silbermann, B. C^*-algebra techniques in numerical analysis. *J. Operator Theory 35*, 2 (1996), 241–280.
[31] Shargorodsky, E. On the level sets of the resolvent norm of a linear operator. *Bull. Lond. Math. Soc. 40* (2008), 493–504.
[32] Strauss, M. The Galerkin method for perturbed self-adjoint operators and applications. *J. Spectr. Theory 4*, 1 (2014), 113–151.
[33] Stummel, F. Diskrete Konvergenz linearer Operatoren. I. *Math. Ann. 190* (1970), 45–92.
[34] Stummel, F. Diskrete Konvergenz linearer Operatoren. II. *Math. Z. 120* (1971), 231–264.
[35] Trefethen, L. N., and Embree, M. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators.* Princeton University Press, 2005.
[36] Vainikko, G. *Funktionalanalysis der Diskretisierungsmethoden.* B. G. Teubner Verlag, Leipzig, 1976.
[37] Weidmann, J. *Lineare Operatoren in Hilberträumen. Teil I.* B. G. Teubner, Stuttgart, 2000.
[38] Wolff, M. P. H. Discrete approximation of unbounded operators and approximation of their spectra. *J. Approx. Theory 113*, 2 (2001), 229–244.

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